HOMOLOGICAL PERTURBATIONS, EQUIVARIANT COHOMOLOGY, AND KOSZUL DUALITY

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Dedicated to the memory of V.K.A.M. Gugenheim

Abstract. Our main objective is to demonstrate how homological perturbation theory (HPT) results over the last 40 years immediately or with little extra work give some of the Koszul duality results that have appeared in the last decade.

Higher homotopies typically arise when a huge object, e. g. a chain complex defining various invariants of a certain geometric situation, is cut to a small model, and the higher homotopies can then be dealt with concisely in the language of sh-structures (strong homotopy structures). This amounts to precise ways of handling the requisite additional structure encapsulating the various coherence conditions. Given e. g. two augmented differential graded algebras A_1 and A_2, an sh-map from A_1 to A_2 is a twisting cochain from the reduced bar construction $\overline{\mathrm{B}}A_1$ of A_1 to A_2 and, in this manner, the class of morphisms of augmented differential graded algebras is extended to that of sh-morphisms. In the present paper, we explore small models for equivariant (co)homology via differential homological algebra techniques including homological perturbation theory which, in turn, is a standard tool to handle sh-structures.

Koszul duality, for a finite type exterior algebra Λ on odd positive degree generators, then comes down to a duality between the category of sh-Λ-modules and that of sh-$\overline{\mathrm{B}}\Lambda$-comodules. This kind of duality relies on the extended functoriality of the differential graded Tor-, Ext-, Cotor-, and Coext functors, extended to the appropriate sh-categories. We construct the small models as certain twisted tensor products and twisted Hom-objects. These are chain and cochain models for the chains and cochains on geometric bundles and are compatible with suitable additional structure.

Contents

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Introduction

The purpose of this paper is to explore suitable small models for equivariant (co)homology via differential homological algebra techniques including, in particular, *homological perturbation theory* (HPT). Our main objective is to demonstrate how HPT results obtained over the last 40 years immediately or with little extra work give some of the Koszul duality results that have appeared in the last decade. Indeed, we will show that these Koszul duality functors yield small models for various differential graded Tor-, Ext-, Cotor-, and Coext functors which, in turn, entail a *conceptual explanation of Koszul duality in terms of the extended functoriality of these functors*. Koszul duality, for a finite type exterior algebra Λ on odd positive degree generators, then comes down to a duality between the category of sh-Λ-modules and that of sh-$\overline{\mathrm{B}}\Lambda$-comodules. Our approach also applies to situations which, to our knowledge, have not been addressed in the literature, e. g. Koszul duality for infinite dimensional groups. The methods we reproduce and newly develop here provide constructive means to explore and handle the requisite additional operations of H_*G on $C_*(X)$ to recover the original action, and categories of sh-modules etc. serve as replacements for various derived categories. We hope to convince the reader that HPT results obtained in the last 40 years are still relevant today and should be better known among mathematicians working in equivariant cohomology.

In general, when standard differential homological algebra techniques yield a huge object calculating some (co)homological invariant and when this object is replaced by an equivalent small model, higher homotopies arise. These higher homotopies reflect a loss of a certain high amount of symmetry which the huge object usually has; thus the huge object is not necessarily "bad". HPT is a standard tool to handle such higher homotopies; we use it here, in particular, to construct twisting cochains and contractions. The notion of twisting cochain in differential homological algebra,

introduced in [6], is intimately related to that of connection in differential geometry, cf. [8], [9], [12], [36], as well as to the *Maurer-Cartan* or *master* equation, cf. [45].

We now give a brief overview of the paper. Section 1 is preliminary in character; is contains a eview of various differential homological algebra techniques and includes in particular a discussion of twisting cochains. In Section 2, we explain the requisite HPT-techniques. In Section 3 we discuss a general notion of duality that arises from an acyclic twisting cochain from a coaugmented differential graded coalgebra C to an augmented differential graded algebra A; the general duality is one between the categories of A-modules and C-comodules. This kind of duality prepares for the conceptual development of Koszul duality later in the paper. The small models for (singular) equivariant G-(co)homology for a topological group G having as homology a strictly exterior algebra in odd degree generators of finite type in the sense that each homogeneous constituent is (necessarily free and) finite-dimensional will be given in Section 4. These models rely on the familiar fact, cf. Theorem 4.1 below that, given a topological group G, the G-equivariant homology and cohomology of a G-space are given by a *differential* Tor and a *differential* Ext, respectively, with reference to the chain algebra of G. This is an Eilenberg-Moore type result. Our approach includes a HPT proof thereof which involves the twisted Eilenberg-Zilber theorem, see Lemma 4.2 below. A variant thereof, given as Theorem 4.1*, describes the G-equivariant cohomology as a suitable Cotor, with reference to an appropriate coalgebra which is a replacement for the cochains on G (which do not inherit a coalgebra structure since in general the dual of a tensor product is not the tensor product of the duals).

In Section 5 we exploit the techniques developed or reproduced in earlier sections to deduce the main Koszul duality results in a conceptual manner. Our constructions of small models extend some of the results in [21, 22, 24] related with *Koszul duality* by placing the latter in the sh-context in the sense isolated in the seminal paper [55] of Stasheff and Halperin; the theory of sh-modules was then exploited in [29, 51, 52] and pushed further in our paper [30]. The 'up to homotopy' interpretation of Koszul duality has been known for a while in the context of operads as well and is also behind e. g. [21]. Yet we believe that our approach in terms of HPT clarifies the meaning and significance of Koszul duality, cf. Section 5 below. In particular, when a group G acts on a space X, even when the induced action of H_*G on H_*X lifts to an action on $C_*(X)$, in general only an sh-action of H_*G on $C_*(X)$ with non-trivial higher terms will recover the geometry of the original action. For example, the ordinary cohomology of a homogeneous space G/K of a compact Lie group G by a closed subgroup K amounts to the K-equivariant cohomology of G, and the corresponding sh-action of H_*K on $C_*(G)$ *cannot* come down to an ordinary (H_*K)-action on $C_*(G)$ since the homogeneous space is compact and finite dimensional. For illustration, let $G = \mathrm{SU}(2)$ and let K be a maximal torus which is just a circle group S^1. For degree reasons, the induced action of $\mathrm{H}_*(S^1)$ on $\mathrm{H}_*(G)$ is trivial and hence lifts to the trivial action of $\mathrm{H}_*(S^1)$ on $C_*(G)$. However this lift cannot recover the geometry of the original S^1-action since if it did, the homology of the homogeneous space G/S^1 would have a free generator in each degree which is impossible since this homogeneous space is just the 2-sphere. This example occurs in [24] (1.5). More details and a class of examples including the one under discussion can be found in Example 7.2 below.

A similar illustration is given in Example 7.1 below. This example serves, in

particular, as an illustration for the notion of twisting cochain. According to a result of Frankel's [20] and Kirwan's [47], a smooth compact symplectic manifold, endowed with a hamiltonian action of a compact Lie group, is equivariantly formal over the reals. Both examples show that the compactness hypothesis is crucial.

What are referred to as *cohomology operations* in [24] is really a *system of higher homotopies encapsulating the requisite sh-action of* H_*G on $C_*(X)$ or $C^*(X)$. Details will be given in Section 5. Suffice it to recall here that an sh-action of an augmented differential graded algebra A on a chain complex V amounts to a twisting cochain from the reduced bar construction $\overline{\mathrm{B}}A$ of A to the differential graded algebra $\mathrm{End}(V)$ of endomorphisms of V. Our HPT-techniques provide the necessary algebraic machinery so that we can handle these higher homotopies and in particular carry out the requisite constructions of e. g. twisting cochains and homotopies of twisting cochains. Koszul duality, for a finite type exterior algebra Λ on odd positive degree generators, comes down to a duality between the category of sh-Λ-modules and that of sh-$\overline{\mathrm{B}}\Lambda$-comodules and, in this context, $\overline{\mathrm{B}}\Lambda$ and the corresponding cofree graded symmetric coalgebra S' are equivalent. For our purposes, the categories of sh-modules and sh-comodules serve as *replacements for various derived categories* exploited in [24] and elsewhere. That sh-structures may be used as replacement for certain derived categories has been observed already in [52].

The duality, then, amounts essentially to the fact that, when $I \to M$ is a relatively injective resolution of an S'-comodule M, cf. e. g. [19] for basic notions of relative homological algebra, this injection is a chain equivalence and that, when $P \to N$ is a relatively projective resolution of a Λ-module N, this projection is a chain equivalence. We explain these chain equivalences under somewhat more general circumstances in Section 3. Concerning Koszul duality, given the Λ-module N and the S'-comodule M, when h and t refer to the two Koszul duality functors, $h(t(N))$ is a generalized projective resolution of N and $t(h(M))$ a generalized injective resolution of M; see (5.3) below for details. Koszul duality reflects the old observation that, for any $k \geq 0$, the non-abelian left derived functor of the k-th exterior power functor in the sense of [14] is the k-th symmetric copower on the suspension (the invariants in the k-th tensor power on the suspension with respect to the symmetric group on k letters).

In Section 6 we offer some unification: For a group G of strictly exterior type, exploiting *isomorphisms* between C_*G and H_*G, between $\overline{\mathrm{B}}C_*G$ and $\mathrm{H}_*(BG)$, and between $\overline{\mathrm{B}}^*C_*G$ and $\mathrm{H}^*(BG)$, in suitably defined categories of sh-algebras and sh-coalgebras as appropriate, we show that the Koszul duality functors are equivalent to certain duality functors on the categories of (C_*G)-modules, $(\overline{\mathrm{B}}^*C_*G)$-modules, and $(\overline{\mathrm{B}}C_*G)$-comodules as appropriate. This equivalence amounts more or less to the extended functoriality of the differential graded Ext-, Tor-, and Cotor-functors in the sense of [29]. We explain this extended functoriality briefly in Section 6 below. The idea behind these isomorphisms goes back to the quoted paper [55] of Stasheff and Halperin.

In Section 7 we illustrate our results with a number of examples from equivariant cohomology. As a final application, in Section 8, we explore a notion of split complex, similar to that introduced in [24] but somewhat more general and adapted to our situation. Our approach includes an interpretation of the splitting homotopy as a generalization of the familiar momentum mapping in symplectic geometry. In particular, using HPT-techniques, we isolate the difference between equivariant

formality and the stronger property that the equivariant cohomology is an induced module over the cohomology of the classifying space.

In the present paper, spectral sequences, one of the basic tools in the literature on Koszul duality and related topics, do not occur explicitly save that we establish the acyclicity of a number of chain complexes by means of an elementary spectral sequence argument. A long time ago, S. Mac Lane pointed out to me that, in various circumstances, spectral sequences are somewhat too weak a tool, on the conceptual as well as computational level. This remark prompted me to develop small models of the kind given below via HPT techniques during the 1980's; the HPT-techniques include refinements of reasoning usually carried out in the literature via spectral sequences. In [31]–[34], I have constructed and exploited suitable small models encapsulating the appropriate sh-structures in the (co)homology of a discrete group, and by means of these small models, I did explicit numerical calculations of group cohomology groups that still today cannot be done by other methods. Equivariant cohomology may be viewed as being part of group cohomology and, in this spirit, the present paper pushes further some of the ideas developed in [30]–[35] and in [44].

In a follow-up paper [41], we have worked out a related approach to equivariant de Rham theory in the framework of suitable relative derived functors, and in [42] we have extended this approach to equivariant de Rham theory relative to a Lie groupoid.

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1. Preliminaries

Let R be a commutative ring with 1, taken henceforth as ground ring and not mentioned any more. Graded objects will be $\mathbb{Z}$-graded; in the applications we will mainly consider only non-negatively or non-positively graded objects, and this will then be indicated.

We will treat chain complexes and cochain complexes on equal footing: We will consider a cochain complex (C^*, d^*) as a chain complex (C_*, d_*) by letting $C_j = C^{-j}$ and $d_* = d^*\colon C_j = C^{-j} \to C^{-j+1} = C_{j-1}$, for $j \in \mathbb{Z}$. An ordinary cochain complex, concentrated in non-negative degrees as a cochain complex, is then a chain complex which is *concentrated in non-positive degrees*. Differential homological algebra terminology and notation will essentially be the same as that in [29] and [46]. We will write the reduced bar and cobar functors as $\overline{\mathrm{B}}$ and $\overline{\Omega}$, respectively, rather than as B and Ω; see e. g. [50] for explicit descriptions. In Section 6 we will reproduce an explicit description of the reduced bar construction. Definitions of the differential Cotor-functor may be found in [19] (p. 206) and in [46] (Chap. 1), and definitions of the differential Tor and Ext functors may be found in [28] (p. 3 and p. 11); see also [18] (p. 7). The *Eilenberg-Koszul* sign convention is in force thoughout. The degree of a homogeneous element x of a graded object is written as $|x|$. Given the chain complexes U and V, the Hom-complex differential on $\mathrm{Hom}(U,V)$ is as usual defined by

$$D\varphi = d\varphi + (-1)^{|\varphi|}\varphi d$$

where φ is a homogeneous, the convention being that $\varphi\colon U_q \to V_p$ has degree $p-q$. The identity morphism on an object is denoted by the same symbol as that object. The suspension operator is written as s; given the chain complex U, the differential on the suspended object sU is determined by the identity $ds + sd = 0$, so that the suspension is a cycle in the corresponding Hom-complex. A *perturbation* of the differential d of a filtered chain complex Z is an operator ∂ on Z that lowers filtration and, moreover, satisfies

$$d\partial + \partial d + \partial\partial = 0,$$

so that $d_\partial = d + \partial$ is a new differential on Z; we shall then write Z_∂ for the new chain complex.

Given a differential graded algebra A, we will refer to a *differential graded* left (or right) A-module more simply as a *left* (or *right*) A-module; we shall denote the categories of differential graded left A-modules by ${}_A\mathrm{Mod}$ and that of differential graded right A-modules by Mod_A. We will write the multiplication map of a differential graded algebra A as $\mu\colon A\otimes A \to A$ and the unit as $\eta\colon R \to A$ when there is a need to spell these structure maps out; more generally, given a right A-module N, we will write the structure map as $\mu\colon N \otimes A \to N$ and, given a left A-module N, we will write the structure map as $\mu\colon A \otimes N \to N$ as well. An augmentation map for the differential graded algebra A is a morphism $\varepsilon\colon A \to R$ of differential graded algebras. A differential graded algebra together with an augmentation map is defined to be *augmented*. Given the augmented differential graded algebra A, with augmentation map $\varepsilon\colon A \to R$, the *augmentation ideal*, written as IA, is the kernel of ε. Given the augmented differential graded algebra A, the *augmentation* filtration $\{\mathrm{F}^n A\}_{n\geq 0}$ given by

$$\mathrm{F}^n A = (IA)^{\otimes n} \ \ (n \geq 0)$$

turns A into a filtered differential graded algebra, and A is *complete* when the canonical map $A \longrightarrow \lim_n A\big/(IA)^n$ into the projective limit $\lim_A A\big/(IA)^n$ is an isomorphism. The augmentation filtration is descending and, at times, it is convenient to write it as $\{\mathrm{F}_n A\}_{n\leq 0}$, where $\mathrm{F}_n A = F^{-n} A$ $(n \leq 0)$. The differential graded algebra A is said to be *connected* when it is non-negative or non-positive and when the unit $\eta\colon R \to A$ is an isomorphism onto the homogeneous degree zero constituent A_0 of A. A connected augmented differential graded algebra is necessarily complete.

Let C be a differential graded coalgebra, the diagonal map being written as $\Delta\colon C \to C \otimes C$ and the counit as $\varepsilon\colon C \to R$. Given a left C-comodule M we will likewise write the structure map as $\Delta\colon M \to C \otimes M$ and, given a right C-comodule M, we will write the structure map as $\Delta\colon M \to M \otimes C$. A *coaugmentation map* for C is a morphism $\eta\colon R \to C$ of differential graded coalgebras and a *coaugmented differential graded coalgebra* is a differential graded coalgebra together with a coaugmentation map. The *coaugmentation* coideal JC is defined to be the cokernel $JC = \mathrm{coker}(\eta)$ of η. Suppose that C is coaugmented. Recall that the counit $\varepsilon\colon C \to R$ and the coaugmentation map determine a direct sum decomposition $C = R \oplus JC$. The *coaugmentation* filtration $\{\mathrm{F}_n C\}_{n\geq 0}$ is as usual given by

$$\mathrm{F}_n C = \ker(C \longrightarrow (JC)^{\otimes(n+1)}) \ \ (n \geq 0)$$

where the unlabelled arrow is induced by some iterate of the diagonal Δ of C. This filtration is ascending and is well known to turn C into a *filtered* coaugmented

differential graded coalgebra; thus, in particular, $\mathrm{F}_0C = R$. We recall that C is said to be *cocomplete* when $C = \cup \mathrm{F}_nC$. The reduced bar construction $\overline{\mathrm{B}}A$ of an augmented differential graded algebra A is well known to be cocomplete. We will refer to a *differential graded* left (right) C-comodule as a *left* (*right*) C-comodule; we shall denote the categories of left C-comodules by ${}_C\mathrm{Comod}$ and that of right C-comodules by Comod_C. The differential graded coalgebra C is said to be connected when it is non-negative or non-positive and when the counit $\varepsilon\colon C \to R$ is an isomorphism from the homogeneous degree zero constituent C_0 of C to R. A connected differential graded coalgebra is necessarily cocomplete. A non-negative connected differential graded coalgebra C is said to be *simply connected* when C_1 is zero. Notice that the cobar construction $\overline{\Omega}C$ on a simply connected differential graded coalgebra C is connected, and so is the cobar construction $\overline{\Omega}C$ on a non-positive differential graded coalgebra C. In the sequel, in a specific construction, coaugmented differential graded coalgebras will be cocomplete throughout but for clarity we will point out explicitly whenever cocompleteness is needed.

Given the two chain complexes U and V, possibly endowed with additional structure, a chain map from U to V possibly preserving additional structure is referred to as a *quasi-isomorphism* provided it induces an isomorphism on homology. A chain complex which, in each degree, is finitely generated, is said to be of *finite type*.

Let $(\{\mathrm{F}_p\}, L) : \ldots \subseteq \mathrm{F}_p \subseteq \mathrm{F}_{p+1} \subseteq \ldots \subseteq L$ be a filtered chain complex. Recall that $(\{\mathrm{F}_p\}, L)$ is said to be *complete* when the canonical map $L \to \lim_p L/\mathrm{F}_p$ into the projective limit $\lim_p L/\mathrm{F}_p$ is an isomorphism of chain complexes, cf. [17] (Section 4) where the terminology P-complete is used and [46] (I.3). For completeness we recall that $(\{\mathrm{F}_p\}, L)$ is said to be *cocomplete* when the canonical map $\lim_p \mathrm{F}_p \to L$ from the injective limit $\lim_p \mathrm{F}_p$ to L is an isomorphism of chain complexes, cf. [17] (Section 4) where the terminology I-complete is used and [46] (§0.5, I §3).

The *skeleton filtration* of the chain complex M is the filtration $\{\mathrm{F}_p(M)\}$ given by

$$(\mathrm{F}_p(M))_n = \begin{cases} M_n, & \text{for} \quad n \leq p, \\ 0, & \text{for} \quad n > p, \end{cases}$$

cf. [46] (§0).

1.1. Twisting cochains and twisted tensor products

Let C be a coaugmented differential graded coalgebra, A an augmented differential graded algebra, M a differential graded left C-comodule, and N a differential graded right A-module. The familiar *cup pairing* $\cup$ turns $\mathrm{Hom}(C, A)$ into a differential graded algebra and $\mathrm{Hom}(M, N)$ into a differential graded *right* $\mathrm{Hom}(C, A)$-module, with structure map of the kind

$$\cup\colon \mathrm{Hom}(M, N) \otimes \mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(M, N). \tag{1.1.1}$$

The cup pairing assigns to $f \otimes h \in \mathrm{Hom}(M, N) \otimes \mathrm{Hom}(C, A)$ the morphism

$$f \cup h\colon M \xrightarrow{\Delta} M \otimes C \xrightarrow{f \otimes h} N \otimes A \xrightarrow{\mu} N.$$

Likewise, the *cap pairing*

$$\cap\colon \mathrm{Hom}(C,A)\otimes N\otimes M \longrightarrow N\otimes M \tag{1.1.2}$$

is given by the assignment to $\varphi\in\mathrm{Hom}(C,A)$ of

$$\varphi\cap\cdot : N\otimes M \xrightarrow{N\otimes\Delta} N\otimes C\otimes M \xrightarrow{N\otimes\varphi\otimes M} N\otimes A\otimes M \xrightarrow{\mu\otimes M} N\otimes M.$$

In the same vein, let N be a differential graded left A-module and M a differential graded right C-comodule. Under these circumstances, the *cap pairing*

$$\cap\colon \mathrm{Hom}(C,A)\otimes M\otimes N \longrightarrow M\otimes N \tag{1.1.3}$$

is given by the assignment to $\varphi\in\mathrm{Hom}(C,A)$ of

$$\varphi\cap\cdot : M\otimes N \xrightarrow{\Delta\otimes N} M\otimes C\otimes N \xrightarrow{M\otimes\varphi\otimes N} M\otimes A\otimes N \xrightarrow{M\otimes\mu} M\otimes N,$$

For example, when $M=C$ and $N=A$, given $\varphi\in\mathrm{Hom}(C,A)$, $c\in C$, and $a\in A$, when $\Delta(c)=\sum c'_j\otimes c''_j$, in view of the Eilenberg-Koszul convention,

$$\varphi\cap(c\otimes a)=\sum(-1)^{|\varphi||c'_j|}c'_j\otimes\varphi(c''_j)a.$$

The cap pairing (1.1.3) turns $M\otimes N$ into a differential graded *left* $\mathrm{Hom}(C,A)$-module and the cap pairing (1.1.2) turns $N\otimes M$ into a differential graded *left* $(\mathrm{Hom}(C,A))^{\mathrm{op}}$-module in the sense that, given homogeneous f, h, w,

$$(f\cup h)\cap w=(-1)^{|f||h|}h\cap f\cap w. \tag{1.1.4}$$

The last identity is an immediate consequence of the identity

$$f\otimes h=(-1)^{|f||h|}(A\otimes h)\circ(f\otimes C)\colon C\otimes C\longrightarrow A\otimes A.$$

Given the differential graded algebra $\mathcal{A}$, the notation $\mathcal{A}^{\mathrm{op}}$ refers here to the *opposite* differential graded algebra. We note that, since the underlying objects are graded, the identity (1.1.4) does *not* say that $N\otimes M$ acquires a right $(\mathrm{Hom}(C,A))$-module structure.

A *twisting cochain* τ from C to A is a homogeneous morphism $\tau\colon C\to A$ of the underlying graded R-modules of degree -1 (i. e. a homogeneous member of $\mathrm{Hom}(C,A)$ of degree -1) such that

$$D\tau=\tau\cup\tau,\ \ \tau\eta=0,\ \ \varepsilon\tau=0,$$

where D refers to the Hom-complex differential on $\mathrm{Hom}(C,A)$.

Let τ be a twisting cochain. Then the operator $\partial^\tau=-(\tau\cap\cdot)$ on $C\otimes A$ (i. e. $\partial^\tau(w)=-\tau\cap w$, $w\in C\otimes A$) is a *perturbation* of the differential on $C\otimes A$, that is, $d+\partial^\tau$ is a differential on $C\otimes A$, and the new differential is compatible with the differential graded left C-comodule and right A-module structures. The resulting differential graded left C-comodule and right A-module is usually written as $C\otimes_\tau A$

and referred to as a *twisted tensor product* or *construction* for A. More generally, given a right C-comodule M and a left A-module N, the twisted tensor product $M \otimes_\tau N$ is defined accordingly. Likewise the operator $\partial^\tau = \tau \cap \cdot$ on $A \otimes C$ (i. e. $\partial^\tau(w) = \tau \cap w$, $w \in A \otimes C$) is a *perturbation* of the differential on $A \otimes C$, that is, $d + \partial^\tau$ is a differential on $A \otimes C$, and the new differential is compatible with the differential graded right C-comodule and left A-module structures. The difference in sign is explained by the structural descriptions of $C \otimes A$ and $A \otimes C$ as $(\mathrm{Hom}(C, A))$- vs. $(\mathrm{Hom}(C, A))^{\mathrm{op}}$-modules explained above.

For later reference we note that, given a differential graded left A-module N, the A-action on N induces the chain map

$$C \otimes_\tau A \otimes N \longrightarrow C \otimes_\tau N \tag{1.1.5}$$

which induces an isomorphism

$$(C \otimes_\tau A) \otimes_A N \longrightarrow C \otimes_\tau N \tag{1.1.6}$$

of chain complexes, cf. [25] (2.6$_*$ Proposition). In fact, cf. [25] (2.4$_*$ Proposition), the twisted differential on $C \otimes_\tau N$ is the unique differential on $C \otimes N$ which makes (1.1.5) into a chain map.

We recall that the twisting cochain τ induces a morphism $\overline{\tau}\colon \overline{\Omega}C \to A$ of augmented differential graded algebras and, when C is cocomplete, a morphism $\overline{\tau}\colon C \to \overline{\mathrm{B}}A$, referred to as the *adjoints* of τ. The reason for this terminology is the fact that the assignment to a twisting cochain $C \to A$ of its adjoint induces a natural bjection between the set $T(C, A)$ of twisting cochains from C to A and the set $\mathrm{Hom}_{\mathrm{alg}}(\overline{\Omega}C, A)$ of morphisms of augmented differential graded algebras and, when C is cocomplete, the assignment to a twisting cochain $C \to A$ of its adjoint induces a natural bjection between $T(C, A)$ and the set $\mathrm{Hom}_{\mathrm{coalg}}(C, \overline{\mathrm{B}}A)$ of morphisms of augmented differential graded coalgebras. Thus the functors $\overline{\Omega}$ and $\overline{\mathrm{B}}$ are adjoint functors between the category of cocomplete coaugmented differential graded coalgebras and that of augmented differential graded algebras. Some connectivity assumptions are necessary here to make this adjointness precise; see e. g. [29] for details.

We will say that the twisting cochain τ is *acyclic* when $C \otimes_\tau A$ is an acyclic complex (and hence an acyclic construction, cf. e. g. [49] for this notion). Subject to mild appropriate additional hypotheses of the kind that C and/or A be projective as R-modules—which will always hold in the paper—the adjoints $\overline{\tau}\colon C \to \overline{\mathrm{B}}A$ and $\overline{\tau}\colon \overline{\Omega}C \to A$ of an acyclic twisting cochain τ are chain equivalences. We denote the universal bar construction twisting cochain by $\tau^{\overline{\mathrm{B}}A}\colon \overline{\mathrm{B}}A \to A$ and the universal cobar construction twisting cochain by $\tau_{\overline{\Omega}C}\colon C \to \overline{\Omega}C$; these are acyclic. Occasionally we write $\tau^{\overline{\mathrm{B}}}$ and $\tau_{\overline{\Omega}}$ rather than $\tau^{\overline{\mathrm{B}}A}$ and $\tau_{\overline{\Omega}C}$, respectively.

Let τ_1 and τ_2 be two twisting cochains from C to A. A *homotopy* of twisting cochains from τ_1 to τ_2, written as $\psi\colon \tau_1 \simeq \tau_2$, is a homogeneous morphism $\psi\colon C \to A$ of degree zero such that

$$D\psi = \tau_1 \cup \psi - \psi \cup \tau_2, \quad \psi\eta = \eta, \quad \varepsilon\psi = \varepsilon \tag{1.1.7}$$

where D refers to the ordinary Hom-complex differential. Here the notation η and ε is slightly abused in the sense that ε denotes the counit of C as well as the

augmentation map of A and that η denotes the unit of A and the coaugmentation map of C.

Let M be a right C-comodule, N a left A-module, and let ψ be a homotopy of twisting cochains from τ_1 to τ_2. Then

$$\psi \cap \cdot \colon M \otimes_{\tau_2} N \longrightarrow M \otimes_{\tau_1} N \tag{1.1.8}$$

is a *morphism* of twisted tensor products. Indeed,

$$(d + \partial^{\tau_1})\,(\psi \cap \cdot\,) + (\psi \cap \cdot\,)\,(d + \partial^{\tau_2}) = (D\psi + \psi \cup \tau_2 - \tau_1 \cup \psi) \cap \cdot\, = 0.$$

Notice when $\psi = \eta\varepsilon$, the morphism $\psi \cap \cdot$ is the identity. Suppose that C is cocomplete. Then (1.1.8) is in fact an *isomorphism*. The inverse map is obtained by a standard procedure: Write $\psi = \eta\varepsilon + \widetilde{\psi}$; the infinite series

$$\psi^{-1} = \eta\varepsilon - \widetilde{\psi} + \widetilde{\psi} \cup \widetilde{\psi} - \widetilde{\psi}^{\cup 3} + \dots \tag{1.1.9}$$

converges since C is cocomplete, and

$$\psi^{-1} \cap \cdot \colon M \otimes_{\tau_1} N \longrightarrow M \otimes_{\tau_2} N \tag{1.1.10}$$

yields the inverse for (1.1.8).

At this stage, one can build a category of twisted tensor products with general notions of morphism and isomorphism; for our purposes, the notion of isomorphism of the kind (1.1.8) suffices.

1.2. Twisted Hom-objects and complete twisted Hom-objects

Let C be a coaugmented differential graded coalgebra, A an augmented differential graded algebra, and $\tau \colon C \to A$ a twisting cochain.

Let N be a differential graded right A-module (we could equally well work with a left A-module) and let δ^τ be the operator on $\mathrm{Hom}(C, N)$ given, for homogeneous f, by $\delta^\tau(f) = (-1)^{|f|} f \cup \tau$. With reference to the filtration induced by the coaugmentation filtration of C, the operator δ^τ is a *perturbation* of the differential d on $\mathrm{Hom}(C, N)$, and we write the perturbed differential on $\mathrm{Hom}(C, N)$ as $d^\tau = d + \delta^\tau$. We will refer to $\mathrm{Hom}^\tau(C, N) = (\mathrm{Hom}(C, N), d^\tau)$ as a *twisted* Hom-*object*. Under suitable circumstances, $\mathrm{Hom}^\tau(C, N)$ calculates the differential graded $\mathrm{Ext}_A(R, N)$.

Lemma 1.2.1. *Suppose that C is cocomplete. Let τ_1 and τ_2 be twisting cochains from C to A and let ψ be a homotopy of twisting cochains from τ_1 to τ_2. The assignment to a (homogeneous) $\phi \in \mathrm{Hom}(C, N)$ of $\phi \cup \psi \in \mathrm{Hom}(C, N)$ yields a morphism, in fact, isomorphism*

$$\mathrm{Hom}^{\tau_1}(C, N) \to \mathrm{Hom}^{\tau_2}(C, N) \tag{1.2.2}$$

of twisted Hom-*objects, with inverse given by the assignment to a (homogeneous) $\phi \in \mathrm{Hom}(C, N)$ of $\phi \cup \psi^{-1} \in \mathrm{Hom}(C, N)$.*

Proof. Indeed,

$$
\begin{aligned}
d^{\tau_2}(\phi \cup \psi) &= (d\phi) \cup \psi + (-1)^{|\phi|}(\phi \cup d\psi + \phi \cup \psi \cup \tau_2) \\
&= (d\phi) \cup \psi + (-1)^{|\phi|}\phi \cup \tau_1 \cup \psi \\
&= (d\phi + (-1)^{|\phi|}\phi \cup \tau_1) \cup \psi \\
&= (d^{\tau_1}(\phi)) \cup \psi
\end{aligned}
$$

whence the assertion. □

One can now build a category of twisted Hom-objects with general notions of morphism and isomorphism; for our purposes, the notion of isomorphism of the kind (1.2.2) suffices.

Consider the canonical injection of $\mathrm{Hom}(C, N)$ into $\mathrm{Hom}(C \otimes A, N)$ which assigns

$$
\Phi_\alpha \colon C \otimes A \to N, \quad \Phi_\alpha(w \otimes a) = \alpha(w)a, \ \ w \in C, \ \ a \in A, \tag{1.2.3}
$$

to $\alpha \in \mathrm{Hom}(C, N)$; this injection plainly identifies $\mathrm{Hom}(C, N)$ with the subspace of (right) A-module morphisms from $C \otimes A$ to N. Furthermore, for homogeneous $\phi \in \mathrm{Hom}(C, A)$,

$$
\Phi_\alpha \circ (\phi \cap \,\cdot\,) = \Phi_{\alpha \cup \phi} \colon C \otimes A \to N, \tag{1.2.4}
$$

where the notation "$\cup$" in the expression $\alpha \cup \phi \in \mathrm{Hom}(C, N)$ refers to the cup pairing (1.1.1). Consequently the assignment to α of Φ_α yields a morphism

$$
\mathrm{Hom}^\tau(C, N) \to \mathrm{Hom}(C \otimes_\tau A, N) \tag{1.2.5}
$$

even of chain complexes, and this morphism identifies $\mathrm{Hom}^\tau(C, N)$ with the subspace $\mathrm{Hom}_A(C \otimes_\tau A, N)$ of differential graded (right) A-module morphisms from $C \otimes_\tau A$ to N.

Let M be a differential graded left A-module. The R-dual M^* of M inherits a canonical differential graded right A-module structure, and the twisted Hom-object $\mathrm{Hom}^\tau(C, M^*)$ is defined. The canonical assignment to $\Phi \colon C \otimes M \to R$ of $\alpha_\Phi \colon C \to M^*$ where

$$
\alpha_\Phi(c)(x) = \Phi(c \otimes x), \ \ c \in C, \ \ x \in M,
$$

yields the adjointness isomorphism

$$
\mathrm{Hom}(C \otimes_\tau M, R) \longrightarrow \mathrm{Hom}^\tau(C, M^*), \tag{1.2.6}
$$

compatible with the perturbed differentials as indicated. Indeed, the perturbation D^τ of the Hom-differential D is given by

$$
D^\tau(\Phi) = (-1)^{|\Phi|+1}\Phi \circ (-\tau \cap \,\cdot\,) = (-1)^{|\Phi|}\Phi \circ (\tau \cap \,\cdot\,)
$$

whereas, for $\alpha \in \mathrm{Hom}(C, M^*)$,

$$
\delta^\tau(\alpha) = (-1)^{|\alpha|}\alpha \cup \tau
$$

whence, indeed, (1.2.6) is compatible with the differentials.

Under these circumstances, when A is of finite type, its dual A^* inherits a coaugmented differential graded coalgebra structure and M^* an obvious left A^*-comodule structure, and the dual $\tau^*\colon A^* \to C^*$ is a twisting cochain. Nota bene: In view of the Eilenberg-Koszul convention, this dual is given by

$$\tau^*(\alpha) = (-1)^{|\alpha|}\alpha\circ\tau,\ \ \alpha\colon A \to R.$$

Furthermore, the canonical morphism of chain complexes

$$C^* \otimes_{\tau^*} M^* \longrightarrow \mathrm{Hom}^\tau(C, M^*) \tag{1.2.7}$$

is a morphism of differential graded C^*-modules and A^*-comodules; it is an isomorphism when C is of finite type. The twisted Hom-object $\mathrm{Hom}^\tau(C, M^*)$ is always defined, though, whether or not A is of finite type. This twisted object is a kind of *completed twisted object* associated with the data A^*, C^*, M^*, and τ^*, where the expression "associated with" is to be interpreted with a grain of salt since $\mathrm{Hom}^\tau(C, M^*)$ involves C and τ rather than merely C^* and τ^* and C and τ are not necessarily determined by C^* and τ^*.

We will now briefly discuss the notion of what we will refer to as a *complete twisted Hom-object*. Let M be a differential graded left C-comodule (we could equally well work with a right C-comodule) and N a differential graded right A-module. Similarly as under the circumstances of (1.2.6), the twisted differential on the right-hand side of the adjointness isomorphism

$$\mathrm{Hom}(N, M^*) \cong \mathrm{Hom}(N\otimes M, R)$$

dual to the twisted differential on $N \otimes_\tau M$ determines a perturbed differential d^τ on $\mathrm{Hom}(N, M^*)$; we write the resulting twisted object as $\mathrm{Hom}^\tau(N, M^*)$ and, similarly as before, refer to it as a *twisted Hom-object*. This situation can be formalized in terms of C^*-modules in the following way: We define a *complete* left C^*-module to be a chain complex $M^\flat$ together with an operation $\rho\colon \mathrm{Hom}(C, M^\flat) \to M^\flat$ which satisfies the obvious associativity and unit constraints. The associativity constraint says that the composite

$$\mathrm{Hom}(C, \mathrm{Hom}(C, M^\flat)) \xrightarrow{\mathrm{Hom}(C,\rho)} \mathrm{Hom}(C, M^\flat) \xrightarrow{\rho} M^\flat$$

equals the composite

$$\mathrm{Hom}(C\otimes C, M^\flat) \xrightarrow{\mathrm{Hom}(\Delta,M^\flat)} \mathrm{Hom}(C, M^\flat) \xrightarrow{\rho} M^\flat$$

where $\mathrm{Hom}(C, \mathrm{Hom}(C, M^\flat))$ and $\mathrm{Hom}(C\otimes C, M^\flat)$ are identified by adjointness in the standard manner; here Δ is the diagonal map of C, and an object and the identity morphism on that object are denoted by the same symbol. Under these circumstances, $\mathrm{Hom}(C, M^\flat)$ is a fortiori a left C^*-module and, when C is of finite type, the operation ρ comes down to an ordinary left C^*-module structure $C^*\otimes M^\flat \to M^\flat$. We denote the category of complete left C^*-modules by ${}_{C^*}\widehat{\mathrm{Mod}}$. The dual M^* of a C-comodule M is a complete C^*-module in an obvious manner.

Let $M^\flat$ be a complete C^*-module, with structure map ρ, and let N be a differential graded right A-module. For $\phi\colon N \to M^\flat$, define $\delta^\tau(\phi)$ to be the composite of

$$\mathrm{Hom}(N, M^\flat) \xrightarrow{\mu^*} \mathrm{Hom}(N \otimes A, M^\flat) \to \mathrm{Hom}(N, \mathrm{Hom}(A, M^\flat))$$

where the unlabelled arrow comes from adjointness, with

$$\mathrm{Hom}(N, \mathrm{Hom}(A, M^\flat)) \xrightarrow{\tau^\sharp} \mathrm{Hom}(N, \mathrm{Hom}(C, M^\flat)) \xrightarrow{\rho_*} \mathrm{Hom}(N, M^\flat);$$

here the notation $\tau^\sharp$ is intended to indicate that that morphism is induced from τ in the obvious manner (but we cannot simply write τ^* since this would conflict with notation established before). The operator δ^τ is a perturbation of the naive differential on $\mathrm{Hom}(N, M^\flat)$, and we will denote the perturbed object by $\mathrm{Hom}^\tau(N, M^\flat)$. Under suitable circumstances, $\mathrm{Hom}(A, M^\flat)$ calculates the differential graded $\mathrm{Tor}_{C^*}(R, M^\flat)$.

The structure map ρ of $M^\flat$ induces a morphism ρ_* from $\mathrm{Hom}(C \otimes N, M^\flat)$ to $\mathrm{Hom}(N, M^\flat)$ which is, in fact, a chain map

$$\rho_*\colon \mathrm{Hom}(C \otimes_\tau N, M^\flat) \to \mathrm{Hom}^\tau(N, M^\flat); \tag{1.2.8}$$

the latter, in turn, passes through a chain map

$$\mathrm{Hom}_{C^*}(C \otimes_\tau N, M^\flat) \to \mathrm{Hom}^\tau(N, M^\flat). \tag{1.2.9}$$

This discussion applies, in particular, to the complete C^*-module M^* relative to the obvious structure coming from the C-comodule structure on M. Moreover, when A is of finite type, its dual A^* inherits a coaugmented differential graded coalgebra structure, the dual $\tau^*\colon A^* \to C^*$ is a twisting cochain, and the canonical morphism of chain complexes

$$A^* \otimes_{\tau^*} M^* \to \mathrm{Hom}^\tau(A, M^*)$$

is an isomorphism of differential graded $\mathrm{Hom}^\tau(A^*, C^*)$-modules. When C is, furthermore, of finite type, the chain map (1.2.8) amounts to

$$\rho_*\colon A^* \otimes_{\tau^*} C^* \otimes M^* \to A^* \otimes_{\tau^*} M^*$$

and (1.2.9) cones down to an isomorphism

$$(A^* \otimes_{\tau^*} C^*) \otimes_{C^*} M^* \to A^* \otimes_{\tau^*} M^*.$$

These are special cases of (1.1.5) and (1.1.6) above.

The twisted Hom-object $\mathrm{Hom}^\tau(A, M^*)$ is always defined, though, whether or not A is of finite type; in a sense explained above, that twisted Hom-object is then a kind of *completed twisted object* associated with the data A^*, C^*, M^*, and τ^*.

1.3. The trivial simplicial object and the total simplicial object

Any object X of a symmetric monoidal category endowed with a diagonal—we will take the categories of spaces, of smooth manifolds, of groups, of vector spaces, of Lie algebras, etc.,—defines two simplicial objects in the category, the *trivial* object which, with an abuse of notation, we still write as X, and the *total object* EX ("total object" not being standard terminology in this generality); the trivial object X has a copy of X in each degree and all simplicial operations are the identity while, for $p \geq 0$, the degree p constituent EX_p of the total object EX is a product of $p+1$ copies of X with the familiar face operations given by omission and degeneracy operations given by insertion; somewhat more formally, *insertion* is here interpreted as an insertion of the diagonal morphism at the appropriate place. When G is a group, this kind of construction leads to the *unreduced homogeneous* bar construction where the term "homogeneous" refers to the fact that the formulation uses the group structure only for the module structure relative to the group and *not* for the simplicial structure. Well known categorical machinery formalizes this situation but we shall not need this kind of formalization. For intelligibility we recall that in [3] the functor which we write as E is denoted by P.

1.4. The geometric resolution and nerve construction

Let G be a topological group. The universal simplicial G-bundle $EG \to NG$ is a special case of the construction spelled out in (1.3) above; indeed, EG acquires a simplicial group structure, the diagonal maps combine to a morphism of simplicial groups $G \to EG$ by means of which EG acquires a simplicial principal right G-space structure, as a simplicial space, $NG = EG/G$, and the projection $EG \to NG$ is the obvious map.

Let X a left G-space. The *simplicial bar construction* $N(G,X)$ is a simplicial space whose geometric realization $|N(G,X)|$ is the Borel construction $EG \times_G X$; with reference to the obvious filtration, $EG \times X$ is, in particular, a free acyclic G-resolution [53] of X in the category of left G-spaces. In particular, when X is a point, $N(G,X)$ comes down to the ordinary *nerve* NG of G, the (lean) realization of which is the ordinary classifying space BG of G. (Within our terminology, it would be more consistent to write $\overline{B}G$ for the classifying space but we will stick to the standard notation BG.) We also note at this stage that, for the sake of clarity, we use the font B for the *algebraic* bar construction. For general X, the *G-equivariant homology* and *cohomology* of X can then be defined as the homology and cohomology of $N(G,X)$. This description works for *any* (co)homology theory, e. g. singular (co)homology or de Rham cohomology: Application of the (co)chain functor to $N(G,X)$ yields a (co)simplicial chain complex the (co)homology of which is the G-equivariant theory. Thus, to define the equivariant theory, there is *no need* to pass through the *Borel construction*, cf. e. g. the discussion in [3]; in particular, this is the correct approach to equivariant de Rham theory since the de Rham functor does not naively apply to the Borel construction. In Theorem 4.1 below we will recall a characterization of equivariant (co)homology as suitable differential Tor- and Ext-functors. When this characterization is taken as the definition, chains, cochains and forms on $N(G,X)$ then appear as an a posteriori object for the calculation of the G-equivariant (co)homology of the G-space X.

It is worthwhile noting that the constructions to be given below apply to the

two-sided constructions at no extra cost: Let E be a right G-space. The obvious action of G on the simplicial space $E \times EG \times X$ is principal, and the *two-sided simplicial bar construction* $N(E,G,X)$ is the base of the resulting simplicial principal G-bundle. We will always suppose that the projection $E \to B = E/G$ is a principal bundle. It is a classical fact that then the canonical projection from $|N(E,G,X)|$ to $E \times_G X$ is a homotopy equivalence which is *compatible with the bundle structures* in the appropriate sense.

1.5. The notion of model

Given a chain complex U, possibly with additional structure (module over a differential graded algebra, comodule over a differential graded coalgebra, twisted object, etc.), we will refer to a chain complex V, possibly endowed with additional structure, as a *model* for U provided there is a chain $U = U_0,\ U_1,\ \dots,\ U_m = V$ of chain complexes, possibly endowed with the additional structure, and a chain of morphisms

$$U_0 \leftarrow U_1,\ \ U_1 \to U_2,\ \ U_2 \leftarrow U_3,\ \ \dots, U_{m-1} \to U_m$$

of chain complexes, each being a quasi-isomorphism and possibly compatible with the additional structure. Under such circumstances, each of these various unlabelled arrows will occasionally be referred to as a *comparison map*.

1.6. Sh-modules and sh-comodules

Given two augmented differential graded algebras A_1 and A_2, in view of the structure of the reduced bar construction, a twisting cochain $\tau\colon \overline{\mathrm{B}}A_1 \to A_2$ has the homogeneous constituents $\tau_j\colon (sIA)^j \to A_2$ $(j \geq 1)$, and a twisting cochain τ having τ_j zero for $j \geq 2$ amounts to an ordinary morphism of differential graded algebras. Non-zero higher terms are an instance of higher homotopies, and a general twisting cochain is an instance of an sh-map, where "sh" stands for "strongly homotopic". Details about the development of these ideas and about the history behind can be found in [40].

Given two augmented differential graded algebras A_1 and A_2, an sh-*map* from A_1 to A_2 is defined to be a twisting cochain $\tau\colon \overline{\mathrm{B}}A_1 \to A_2$. Likewise, given two coaugmented differential graded coalgebras C_1 and C_2, an sh-*map* from C_1 to C_2 is a twisting cochain $\tau\colon C_1 \to \overline{\Omega}C_2$.

Let C be a coaugmented differential graded coalgebra, A an augmented differential graded algebra, and let $\tau\colon C \to A$ be an acyclic twisting cochain. We will refer to a left (right) A'-module N, for some augmented differential graded algebra A', together with a twisting cochain $\tau_{A'}\colon C \to A'$, as a *left* (*right*) sh-*module over* A. An ordinary A-module is an sh-module over A in an obvious manner. Thus the notion of sh-module over A extends that of ordinary A-module. When $(N, \tau_{A'})$ is an sh-*module over* A, the chain complex N acquires an $(\overline{\Omega}C)$-module structure via the adjoint $\overline{\tau}_{A'}\colon \overline{\Omega}C \to A'$, and we may in particular take $A' = \overline{\Omega}C$ and $\tau_{A'} = \tau_{\overline{\Omega}C}$; thus requiring the twisting cochain $\tau_{A'}\colon C \to A'$ to be acyclic yields an equivalent notion of sh-module over A. The given definition allows for more flexibility, though. Likewise a *left* (*right*) sh-*comodule over* C is a left (right) C'-comodule M, for some *cocomplete* coaugmented differential graded coalgebra C', together with a twisting cochain $\tau^{C'}\colon C' \to A$. An ordinary C-comodule is an sh-comodule over C in an

obvious manner. Thus the notion of sh-comodule over C extends that of ordinary C-comodule. When $(M,\tau^{C'})$ is an sh-*comodule over* C, the chain complex M acquires a $(\overline{\mathrm{B}}A)$-comodule structure via the adjoint $\overline{\tau}^{C'}\colon C' \to \overline{\mathrm{B}}A$, and we may in particular take $C' = \overline{\mathrm{B}}A$ and $\tau^{C'} = \tau^{\overline{\mathrm{B}}A}$; again requiring the twisting cochain $\tau^{C'}\colon C' \to A$ to be acyclic yields an equivalent notion of sh-comodule over C, and the given definition allows for more flexibility.

Given two sh-modules $(N_1,\tau_{A_1'})$ and $(N_2,\tau_{A_2'})$ over A, we *define* an *sh-morphism* from $(N_1,\tau_{A_1'})$ to $(N_2,\tau_{A_2'})$ to be a morphism $N_1 \to N_2$ of differential graded $(\overline{\Omega}C)$-modules where the $(\overline{\Omega}C)$-module structures on N_1 and N_2 are induced from the twisting cochains $\tau_{A_1'}$ and $\tau_{A_2'}$, respectively. Denote the resulting category by ${}_A\mathrm{Mod}^\infty$ or Mod_A^∞, according as whether left or right modules are involved. An ordinary A-module morphism is an sh-morphism in an obvious manner. Thus the notion of sh-morphism extends that of ordinary A-module morphism.

Likewise, given two sh-comodules $(M_1,\tau^{C_1'})$ and $(M_2,\tau^{C_2'})$ over C, we *define* an *sh-morphism* from $(M_1,\tau^{C_1'})$ to $(M_2,\tau^{C_2'})$ to be a morphism $M_1 \to M_2$ of differential graded $(\overline{\mathrm{B}}A)$-comodules where the $(\overline{\mathrm{B}}A)$-comodule structures on M_1 and M_2 are induced from the twisting cochains $\tau^{C_1'}$ and $\tau^{C_2'}$, respectively. Denote the resulting category by ${}_C\mathrm{Comod}^\infty$ or Comod_C^∞, according as whether left- or right comodules are under discussion. An ordinary morphism of C-comodules is an sh-morphism in an obvious manner. Thus the notion of sh-morphism extends that of ordinary morphism of C-comodules.

As before, let $\tau\colon C \to A$ be an acyclic twisting cochain. Further, let C' be a coaugmented differential graded coalgebra, A' an augmented differential graded algebra, and let $\zeta_{A'}\colon C \to A'$ and $\zeta^{C'}\colon C' \to A$ be acyclic twisting cochains. The following is straightforward.

Proposition 1.6.1$_*$. *The assignment to a left A'-module N of the left sh-module $(N,\zeta_{A'})$ over A is a functor from ${}_{A'}\mathrm{Mod}$ to ${}_A\mathrm{Mod}^\infty$ and, likewise, the assignment to a left C'-comodule M of the left sh-comodule $(M,\zeta^{C'})$ over C is a functor from ${}_{C'}\mathrm{Comod}$ to ${}_C\mathrm{Comod}^\infty$.* $\square$

2. Homological perturbations

Homological perturbation theory (HPT) is concerned with transferring various kinds of algebraic structure through a homotopy equivalence. Historical comments about the development of HPT may be found in [40] and [43]. The basic reason why HPT works is the old observation that an exact sequence of chain complexes which splits as an exact sequence of graded modules and which has a contractible quotient necessarily splits in the category of chain complexes [13] (2.18).

Here is an essential piece of machinery.

Definition 2.1. A *contraction*

$$(M \underset{\nabla}{\overset{\pi}{\leftrightarrows}} N, h) \tag{2.1.1}$$

of chain complexes consists of chain complexes N and M, chain maps $\pi\colon N \to M$ and $\nabla\colon M \to N$, and a morphism $h\colon N \to N$ of the underlying graded modules of degree 1; these data are required to satisfy the identites

$$\pi\nabla = \mathrm{Id}, \quad Dh = \nabla\pi - \mathrm{Id}, \tag{2.1.2}$$

$$\pi h = 0, \quad h\nabla = 0, \quad hh = 0. \tag{2.1.3}$$

We will then say that N *contracts onto* M. If furthermore, N and M are filtered chain complexes, and if π, ∇ and h are filtration preserving, the contraction is said to be *filtered*. The requirements (2.1.3) are referred to as *annihilation properties* or *side conditions*.

The notion of contraction was introduced in §12 of [16]; it is among the basic tools in homological perturbation theory, cf. [44] and the literature there.

Remark 2.1.4. It is well known that the side conditions (2.1.3) can always be achieved. This fact relies on the standard observation that a chain complex is contractible if and only if it is isomorphic to a cone, cf. [46] (IV.1.5). This observation is, by the way, related with that of Dold's quoted above. Under the present circumstances, given data of the kind (2.1.1) such that the identities (2.1.2) hold but not necessarily the side conditions (2.1.3), the operator

$$\widetilde{h} = (\mathrm{Id} - \nabla\pi)h(\mathrm{Id} - \nabla\pi)d(\mathrm{Id} - \nabla\pi)h(\mathrm{Id} - \nabla\pi)$$

satisfies the requirements (2.1.2) and (2.1.3), with $\widetilde{h}$ instead of h; when h already satisfies (2.1.3), $\widetilde{h}$ coincides with h.

The argument in [29] (4.1) establishes the following:

Proposition 2.2$_*$. *Let C and C' be coaugmented differential graded coalgebras, let $\left(C \underset{\nabla}{\overset{\pi}{\leftrightarrows}} C', h\right)$ be a contraction of chain complexes where ∇ is a morphism of differential graded coalgebras, and let A be a complete augmented differential graded algebra. Given a twisting cochain $\sigma\colon C \to A$, there is a unique twisting cochain $\xi\colon C' \to A$ with $\xi\nabla = \sigma$ and $\xi h = 0$. This twisting cochain is given by the inductive formula*

$$(2.2.1_*) \qquad \xi = \sigma\pi - (\xi \cup \xi)h$$

and is natural in terms of the data. $\square$

Somewhat more explicitly, the formula (2.2.1$_*$) means that $\xi = \xi_1 + \xi_2 + \dots$ where

$$\xi_1 = \sigma\pi, \;\; \xi_p = -\sum_{i+j=p} (\xi_i \cup \xi_j)h \quad (p > 1).$$

For any $n \geq 1$, the composite of ξ_p with the canonical projection $A \to A\big/(IA)^n$ is non-zero for only finitely many p whence the construction yields a twisting cochain with values in the projective limit $\lim_n A\big/(IA)^n$. Since A is supposed to be complete, the construction actually yields a twisting cochain with values in A itself.

The dual statement reads as follows.

Proposition 2.2*. *Let A and A' be augmented differential graded algebras, let $\left(A \underset{\nabla}{\overset{\pi}{\leftrightarrows}} A', h\right)$ be a contraction of chain complexes where π is a morphism of differential graded algebras, and let C be a cocomplete coaugmented differential graded coalgebra. Given a twisting cochain $\sigma\colon C \to A$, there is a unique twisting cochain $\xi\colon C \to A'$ with $\pi\xi = \sigma$ and $h\xi = 0$. This twisting cochain is given by the inductive formula*

$$(2.2.1^*) \qquad \xi = \nabla\sigma - h(\xi \cup \xi)$$

and is natural in terms of the data. □

Here, for each $n \geq 1$, on the degree n coaugmentation filtration constituent $\mathrm{F}_n C$ of C, only finitely many of the recursive terms of ξ are non-zero. Since C is cocomplete, each element of C actually lies in some $\mathrm{F}_n C$ whence, given a particular element c of C, the convergence of $\xi(c)$ is naive.

Lemma 2.3. [Perturbation lemma] *Let* $(M \underset{\nabla}{\overset{g}{\leftrightarrows}} N, h)$ *be a filtered contraction, let* ∂ *be a perturbation of the differential on* N*, and let*

$$\mathcal{D} = \sum_{n\geq 0} g\partial(h\partial)^n\nabla = \sum_{n\geq 0} g(\partial h)^n\partial\nabla \tag{2.3.1}$$

$$\nabla_\partial = \sum_{n\geq 0} (h\partial)^n\nabla \tag{2.3.2}$$

$$g_\partial = \sum_{n\geq 0} g(\partial h)^n \tag{2.3.3}$$

$$h_\partial = \sum_{n\geq 0} (h\partial)^n h = \sum_{n\geq 0} h(\partial h)^n. \tag{2.3.4}$$

If the filtrations on M *and* N *are complete, these infinite series converge,* $\mathcal{D}$ *is a perturbation of the differential on* M *and, if we write* N_∂ *and* $M_\mathcal{D}$ *for the new chain complexes,*

$$(M_\mathcal{D} \underset{\nabla_\partial}{\overset{g_\partial}{\leftrightarrows}} N_\partial, h_\partial) \tag{2.3.5}$$

constitute a new filtered contraction that is natural in terms of the given data.

Proof. Details may be found in [7] and in Lemmata 3.1 and 3.2 of [25]. □

Remark 2.4. Given an arbitrary augmented differential graded algebra A, the construction in [50] (2.14 Proposition, 2.15 Corollary) yields a contraction

$$(\overline{\Omega \mathrm{B}} A \underset{\overline{\Omega}\tau}{\overset{\nabla}{\leftrightarrows}} A, h)$$

that is natural in A in such a way that, by construction, $\overline{\Omega}\tau$ is a morphism of differential graded algebras. See also [46] (II.4.4 Theorem p. 148). The corresponding result for a connected coaugmented differential graded coalgebra is given in [46] (II.4.5 Theorem p. 148). The special cases for a connected algebra A and a simply connected coalgebra C can be found already in [15].

3. Duality

Let C be a coaugmented differential graded coalgebra, A an augmented differential graded algebra, and $\tau\colon C \to A$ an acyclic twisting cochain. Suppose there be given an explicit contracting homotopy $s\colon C\otimes_\tau A \to C\otimes_\tau A$ on $C\otimes_\tau A$ as well, so that $(R \underset{\eta}{\overset{\varepsilon}{\leftrightarrows}} C\otimes_\tau A, s)$ is a contraction, i. e. $Ds = \mathrm{Id}-\eta\varepsilon$, $ss=0$, etc. In view of Remark 2.1.4, once a contracting for $C\otimes_\tau A$ is given, the requirement $ss=0$ can always be arranged for. With a slight abuse of notation, we will denote the corresponding contracting homotopy of $A\otimes_\tau C$ by s as well. These contracting homotopies will be used in Theorems 3.5_* and 3.5^* below.

We suppose that the cohomology of C, that is, the homology of C^*, is of finite type and that, likewise, the homology of A^* is of finite type as well. These hypotheses are not independent, see Remark 3.6 below. Consider the functors

$$t\colon {}_A\mathrm{Mod} \longrightarrow {}_C\mathrm{Comod}, \quad t(N) = C\otimes_\tau N \tag{3.1$_*$}$$

$$h\colon {}_C\mathrm{Comod} \longrightarrow {}_A\mathrm{Mod}, \quad h(M) = A\otimes_\tau M. \tag{3.2$_*$}$$

Unfortunately the notation h conflicts with the earlier notation for a contracting homotopy but, for intelligibility, we prefer to keep the notation h for both objects; in the sequel, which one is intended will always be clear from the context.

Given the left C-comodule M, the composite

$$M \longrightarrow C\otimes_\tau A\otimes_\tau M = t(h(M)) \tag{3.3$_*$}$$

of the comodule structure map $M \longrightarrow C\otimes M$ with the canonical injection

$$C\otimes M \to C\otimes A\otimes M$$

is a morphism of left C-comodules that is natural in M. Henceforth the numbering (3.3_*) will also refer to the corresponding natural transformation $\mathcal{I}\to th$ where $\mathcal{I}$ refers to the identity functor.

Likewise, given the left A-module N, the composite

$$h(t(N)) = A\otimes_\tau C\otimes_\tau N \longrightarrow N \tag{3.4$_*$}$$

of the canonical projection $A\otimes C\otimes N \to A\otimes N$ with the A-module structure map $A\otimes N \to N$ of N is a morphism of left A-modules that is natural in N. Henceforth the numbering (3.4_*) will also refer to the corresponding natural transformation $ht \to \mathcal{I}$.

The functors t and h are chain homotopy inverse to each other in a very precise sense which we now explain.

Let $\mathcal{F}\colon {}_C\mathrm{Comod} \to {}_C\mathrm{Comod}$ and $\mathcal{G}\colon {}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$ be functors, and denote the identity functor by $\mathcal{I}$. We define a *left C-comodule functor contraction of $\mathcal{F}$ to the identity* to be a contraction

$$(\mathcal{I} \underset{\nabla}{\overset{\pi}{\leftrightarrows}} \mathcal{F}, s);$$

here ∇, π, s are natural transformations in such a way that, for every left C-comodule M, the data

$$(M \underset{\nabla_M}{\overset{\pi_M}{\leftrightarrows}} \mathcal{F}(M), s_M)$$

constitute a contraction of chain complexes with ∇_M a morphism of C-comodules. Likewise we define a *left A-module functor contraction of $\mathcal{G}$ to the identity* to be a contraction

$$(\mathcal{I} \overset{\pi}{\underset{\nabla}{\leftrightarrows}} \mathcal{G}, s);$$

here ∇, π, s are natural transformations in such a way that, for every left A-module N, the data

$$(N \overset{\pi_N}{\underset{\nabla_N}{\leftrightarrows}} \mathcal{G}(N), s_N)$$

constitute a contraction of chain complexes with π_N a morphism of A-modules. Accordingly we define the notion of *right C-comodule functor contraction to the identity* and that of *right A-module functor contraction to the identity*.

Theorem 3.5$_*$. *The functors t and h are chain homotopy inverse to each other in the sense of* (1) *and* (2) *below:*

(1) *The natural transformation* (3.3$_*$) *of left C-comodules extends to a left C-comodule contraction to the identity*

$$(\mathcal{I} \overset{\pi}{\underset{(3.3_*)}{\leftrightarrows}} th, s)$$

*of the endofunctor th on ${}_C$*Comod*;*

(2) *the natural transformation* (3.4$_*$) *of left A-modules extends to a left A-module contraction to the identity*

$$(\mathcal{I} \overset{(3.4_*)}{\underset{\nabla}{\leftrightarrows}} ht, s)$$

*of the endofunctor ht on ${}_A$*Mod.

Proof. Let M be a left C-comodule. The projection

$$\varepsilon_M = \varepsilon \otimes \varepsilon \otimes \mathrm{Id}\colon t(h(M)) = C \otimes_\tau A \otimes_\tau M \to M \tag{3.5.1$_*$}$$

is a chain map (beware: not a morphism of C-comodules) that is a retraction for (3.3$_*$). Furthermore, ε_M and (3.3$_*$) are also chain maps when $C \otimes_\tau A \otimes M$ is substituted for $C \otimes_\tau A \otimes_\tau M$, and the morphism

$$s^0_M = s \otimes \mathrm{Id}\colon C \otimes_\tau A \otimes M \longrightarrow C \otimes_\tau A \otimes M \tag{3.5.2$_*$}$$

yields a chain homotopy between the identity and the composite (3.3$_*$) ε_M. Thus the data

$$(M \overset{\varepsilon_M}{\underset{(3.3_*)}{\leftrightarrows}} C \otimes_\tau A \otimes M, s^0_M) \tag{3.5.3$_*$}$$

constitute a contraction. If the side conditions (2.1.3) are not satisfied we can modify s^0_M if need be, cf. Remark 2.1.4, and we suppose that this has already been arranged for. We do not indicate this in notation. By construction,

$$\varepsilon_M s^0_M = 0, \ \ s^0_M (3.3_*) = 0. \tag{3.5.4$_*$}$$

Write the differential on $C \otimes_\tau A \otimes_\tau M$ as $d + \partial^\tau$ where d refers to the differential on $C \otimes_\tau A \otimes M$. Relative to the filtration of $C \otimes_\tau A \otimes_\tau M$ induced by the skeletal filtration of M, the operator ∂^τ lowers filtration, and this filtration is complete. Application of the perturbation lemma (Lemma 2.3) yields a new contraction

$$(M \underset{(3.3_*)}{\overset{\varepsilon_M}{\leftrightarrows}} C \otimes_\tau A \otimes_\tau M, s_M). \tag{3.5.5$_*$}$$

In view of (3.5.4$_*$), the perturbation modifies only the homotopy s^0_M. The morphisms (3.3$_*$), ε_M and s_M are plainly natural in M.

Likewise, let N be a left A-module. The injection

$$\eta_N = \eta \otimes \eta \otimes \mathrm{Id}\colon N \longrightarrow A \otimes_\tau C \otimes_\tau N = h(t(N)) \tag{3.5.6$_*$}$$

is a chain map (beware: not a morphism of A-modules) that is a section for (3.4$_*$). Furthermore, η_N and (3.4$_*$) are also chain maps when $A \otimes_\tau C \otimes N$ is substituted for $A \otimes_\tau C \otimes_\tau N$, and the morphism

$$s^0_N = s \otimes \mathrm{Id}\colon A \otimes_\tau C \otimes N \longrightarrow A \otimes_\tau C \otimes N \tag{3.5.7$_*$}$$

yields a chain homotopy between the identity and the composite $\eta_N(3.4_*)$. Thus the data

$$(N \underset{\eta_N}{\overset{(3.4_*)}{\leftrightarrows}} A \otimes_\tau C \otimes N, s^0_N) \tag{3.5.8$_*$}$$

constitute a contraction. Again, if the side conditions (2.1.3) are not satisfied we can modify s^0_N if need be, cf. Remark 2.1.4, and we suppose that this has already been arranged for. We do not indicate this in notation. By construction,

$$s^0_N \eta_N = 0, \ \ (3.3_*) s^0_N = 0. \tag{3.5.9$_*$}$$

Write the differential on $A \otimes_\tau C \otimes_\tau N$ as $d + \partial^\tau$ where d refers to the differential on $A \otimes_\tau C \otimes N$. Relative to the filtration of $A \otimes_\tau C \otimes_\tau N$ coming from the skeletal filtration of $A \otimes_\tau C$, the operator ∂^τ lowers filtration, and the filtration is complete. Application of the perturbation lemma (Lemma 2.3) yields a new contraction

$$(N \underset{\eta_N}{\overset{(3.4_*)}{\leftrightarrows}} A \otimes_\tau C \otimes_\tau N, s_N) \tag{3.5.10$_*$}$$

In view of (3.5.9$_*$), the perturbation modifies only the homotopy s^0_N. The morphisms (3.4$_*$), η_N and s_N are plainly natural in N. $\square$

The theorem implies the following: Suppose that C and A satisfy certain mild arithmetical hypotheses such as e. g. that they are projective over the ground ring as graded modules. Then, when C is connected, the twisted object $t(h(M))$ serves as a replacement for a relatively injective resolution of M in the category of left C-comodules, that is, given the right C-comodule M', the homology of the cotensor product

$$M' \square_C t(h(M)) = M' \square_C (C \otimes_\tau A \otimes_\tau M) \cong M' \otimes_\tau A \otimes_\tau M$$

yields the differential graded $\operatorname{Cotor}^C(M',M)$. When C is, furthermore, cocomplete, the twisted object $h(t(N))$ serves as a replacement for a relatively projective resolution of N in the category of left A-modules, that is, given the right A-comodule N', the homology of the tensor product

$$N' \otimes_A h(t(N)) = N' \otimes_A (A \otimes_\tau C \otimes_\tau N) \cong N' \otimes_\tau C \otimes_\tau N$$

yields the differential graded $\operatorname{Tor}_A(N',N)$. In particular, the functor t assigns to a left A-module N a twisted tensor product which calculates the differential graded $\operatorname{Tor}^A(R,N)$, and the functor h assigns to a left C-comodule M a twisted tensor product which, when C is connected, calculates the differential graded $\operatorname{Cotor}^C(R,M)$. The reason for the connectedness hypothesis is that, in the definition of Cotor, the injective resolution must be assembled by the product and not by the coproduct that is used in the definition of both $\overline{\mathrm{B}}$ and $\overline{\Omega}$.

The dual C^* of C is an augmented differential graded algebra. We remind the reader that ${}_{C^*}\widehat{\mathrm{Mod}}$ refers to the category of C^*-modules which are complete in a sense explained in (1.2.4) above. Relative to the differential graded algebra C^*, the functor

$$t^*\colon \mathrm{Mod}_A \to {}_{C^*}\widehat{\mathrm{Mod}}, \quad t^*(N) = \operatorname{Hom}^\tau(C,N) \tag{3.1*}$$

assigns to a right A-module N the twisted Hom-object $t^*(N)$ which calculates the differential graded $\operatorname{Ext}_A(R,N)$, and the functor

$$h^*\colon {}_{C^*}\widehat{\mathrm{Mod}} \to \mathrm{Mod}_A, \quad h^*(M^\flat) = \operatorname{Hom}^\tau(A,M^\flat) \tag{3.2*}$$

assigns to a (complete) left C^*-module $M^\flat$ the twisted Hom-object $h^*(M^\flat)$ which, since the homology of C^* is of finite type, calculates the differential graded $\operatorname{Tor}^{C^*}(R,M^\flat)$. When A is itself of finite type, A^* is a differential graded coalgebra, the dual $\tau^*\colon A^* \to C^*$ (given by $\tau^*(\alpha) = (-1)^{|\alpha|}\alpha\circ\tau$) is a twisting cochain, and the obvious morphism is an isomorphism

$$A^* \otimes_{\tau^*} M^\flat \to \operatorname{Hom}^\tau(A,M^\flat)$$

of (differential graded) left A^*-comodules. Whether or not A is of finite type, given the complete left C^*-module $M^\flat$, the composite of the structure map $\operatorname{Hom}(C,M^\flat)\to M^\flat$ with the morphism

$$\operatorname{Hom}(C,\operatorname{Hom}(A,M^\flat)) \to \operatorname{Hom}(C,M^\flat)$$

induced by the evaluation map

$$\operatorname{Hom}(A,M^\flat) \to M^\flat, \quad \phi \mapsto \phi(1),$$

yields a morphism

$$M^\flat \leftarrow \operatorname{Hom}^\tau(C\otimes_\tau A, M^\flat) \cong \operatorname{Hom}^\tau(C,\operatorname{Hom}^\tau(A,M^\flat)) = t^*(h^*(M^\flat)) \tag{3.3*}$$

of left C^*-modules which is also a chain equivalence that is natural in $M^\flat$, and $t^*(h^*(M^\flat))$ is relatively projective when C is of finite type. Likewise, let N be a

right A-module. The composite of the adjoint $N \to \mathrm{Hom}(A,N)$ of the structure map $N \otimes A \to N$ with the canonical injection

$$\varepsilon^*\colon \mathrm{Hom}(A,N) \longrightarrow \mathrm{Hom}(A,\mathrm{Hom}(C,N))$$

induced by the counit ε of C yields a morphism

$$(3.4^*) \qquad N \longrightarrow \mathrm{Hom}^\tau(A \otimes_\tau C, N) \cong \mathrm{Hom}^\tau(A, \mathrm{Hom}^\tau(C,N)) = h^*(t^*(N))$$

in the category of right A-modules which is also a chain equivalence that is natural in N.

We define a *complete left C^*-module functor contraction to the identity* of the functor $\mathcal{G}\colon {}_{C^*}\widehat{\mathrm{Mod}} \to {}_{C^*}\widehat{\mathrm{Mod}}$ to be a contraction

$$(\mathcal{I} \underset{\nabla}{\overset{\pi}{\leftrightarrows}} \mathcal{G}, s)$$

as defined before for A-modules save that now the constituents of the contraction refer to complete C^*-modules. Henceforth the numberings (3.3^*) and (3.4^*) will also refer to the corresponding natural transformations.

Theorem 3.5*. *The functors t^* and h^* are chain homotopy inverse to each other in the sense of* (1) *and* (2) *below:*

(1) *The natural transformation* (3.3^*) *of complete left C^*-modules extends to a complete left C^*-module contraction to the identity*

$$(\mathcal{I} \underset{\nabla}{\overset{(3.3^*)}{\leftrightarrows}} t^*h^*, s)$$

of the endofunctor t^*h^* on ${}_{C^*}\widehat{\mathrm{Mod}}$;

(2) *the natural transformation* (3.4^*) *of A-modules extends to a contraction to the identity*

$$(\mathcal{I} \underset{(3.4^*)}{\overset{\pi}{\leftrightarrows}} h^*t^*, s)$$

of the endofunctor h^*t^* on Mod_A.

Proof. Let $M^\flat$ be a complete left C^*-module. The obvious injection

$$(3.5.1^*) \qquad \eta_{M^\flat}\colon M^\flat \longrightarrow \mathrm{Hom}^\tau(C \otimes_\tau A, M^\flat) \cong \mathrm{Hom}^\tau(C, \mathrm{Hom}^\tau(A, M^\flat)) = t^*(h^*(M^\flat))$$

is a chain map (beware: not a morphism of C^*-modules) that is a section for (3.3^*). Furthermore, $\eta_{M^\flat}$ and (3.3^*) are also chain maps when $\mathrm{Hom}(C \otimes_\tau A, M^\flat)$ is substituted for $\mathrm{Hom}^\tau(C \otimes_\tau A, M^\flat)$, and the morphism

$$(3.5.2^*) \qquad s^0_{M^\flat}\colon \mathrm{Hom}(C \otimes_\tau A, M^\flat) \longrightarrow \mathrm{Hom}(C \otimes_\tau A, M^\flat), \ \ s^0_{M^\flat}(\beta) = (-1)^{|\beta|}\beta \circ s$$

yields a chain homotopy between the identity and the composite $\eta_{M^\flat}(3.3^*)$. Thus the data

$$(3.5.3^*) \qquad (M^\flat \underset{\eta_{M^\flat}}{\overset{(3.3^*)}{\leftrightarrows}} \mathrm{Hom}(C \otimes_\tau A, M^\flat), s^0_{M^\flat})$$

constitute a contraction. If the side conditions (2.1.3) are not satisfied we can modify $s^0_{M^\flat}$ if need be, cf. Remark 2.1.4, and we suppose that this has already been arranged for. We do not indicate this in notation. By construction,

$$s^0_{M^\flat}\eta_{M^\flat} = 0,\ \ (3.3^*)s^0_{M^\flat} = 0. \tag{3.5.4*}$$

Write the differential on $\mathrm{Hom}^\tau(C\otimes_\tau A, M^\flat)$ as $d+\delta^\tau$ where d refers to the differential on $\mathrm{Hom}(C\otimes_\tau A, M^\flat)$. Relative to the filtration of $\mathrm{Hom}^\tau(C\otimes_\tau A, M^\flat)$ induced by the skeletal filtration of $M^\flat$, the operator δ^τ lowers filtration, and the filtration of $\mathrm{Hom}^\tau(C\otimes_\tau A, M^\flat)$ is complete. Application of the perturbation lemma (Lemma 2.3) yields a new contraction

$$(M^\flat \underset{\eta_{M^\flat}}{\overset{(3.3^*)}{\rightleftarrows}} \mathrm{Hom}^\tau(C\otimes_\tau A, M^\flat), s_{M^\flat}) \tag{3.5.5*}$$

In view of (3.5.4*), the perturbation modifies only the homotopy $s^0_{M^\flat}$. The morphisms (3.3*), $\eta_{M^\flat}$ and $s_{M^\flat}$ are plainly natural in M.

Likewise, let N be a right A-module. The obvious projection

$$\varepsilon_N\colon h^*(t^*(N)) = \mathrm{Hom}^\tau(A, \mathrm{Hom}^\tau(C,N)) \longrightarrow N \tag{3.5.6*}$$

is a chain map (beware: not a morphism of A-modules) that is a retraction for (3.4*). Furthermore, ε_N and (3.4*) are also chain maps when $\mathrm{Hom}^\tau(A, \mathrm{Hom}(C,N))$ is substituted for $\mathrm{Hom}^\tau(A, \mathrm{Hom}^\tau(C,N))$. By adjointness,

$$\mathrm{Hom}^\tau(A, \mathrm{Hom}(C,N)) \cong \mathrm{Hom}(A\otimes_\tau C, N)$$

and the morphism

$$s^0_N\colon \mathrm{Hom}(A\otimes_\tau C, N) \longrightarrow \mathrm{Hom}(A\otimes_\tau C, N),\ \ s^0_N(\beta) = (-1)^{|\beta|}\beta\circ s \tag{3.5.7*}$$

yields a chain homotopy between the identity and the composite $(3.4^*)\varepsilon_N$. Thus the data

$$(N \underset{\varepsilon_N}{\overset{(3.4^*)}{\rightleftarrows}} \mathrm{Hom}(A\otimes_\tau C, N), s^0_N)$$

constitute a contraction and, for the sake of clarity of exposition, we write this contraction as

$$(N \underset{\varepsilon_N}{\overset{(3.4^*)}{\rightleftarrows}} \mathrm{Hom}^\tau(A, \mathrm{Hom}(C,N)), s^0_N), \tag{3.5.8*}$$

with a slight abuse of the notation s^0_N. Again, if the side conditions (2.1.3) are not satisfied we can modify s^0_N if need be, cf. Remark 2.1.4, and we suppose that this has already been arranged for. We do not indicate this in notation. By construction,

$$\varepsilon_N s^0_N = 0,\ \ s^0_N(3.3^*) = 0. \tag{3.5.9*}$$

Write the differential on $\mathrm{Hom}^\tau(A, \mathrm{Hom}^\tau(C,N))$ as $d+\delta^\tau$ where d refers to the differential on $\mathrm{Hom}^\tau(A, \mathrm{Hom}(C,N))$. Relative to the filtration of

$$\mathrm{Hom}(A, \mathrm{Hom}(C,N)) \cong \mathrm{Hom}(A\otimes C, N)$$

coming from the skeletal filtration of $A \otimes C$, the operator δ^τ lowers filtration, and the filtration of $\mathrm{Hom}(A, \mathrm{Hom}(C, N))$ is complete. Application of the perturbation lemma (Lemma 2.3) yields a new contraction

$$(N \underset{\varepsilon_N}{\overset{(3.4^*)}{\leftrightarrows}} \mathrm{Hom}^\tau(A, \mathrm{Hom}^\tau(C, N)), s_N) \tag{3.5.10*}$$

In view of (3.5.9*), the perturbation modifies only the homotopy s_N^0. The morphisms η_N and s_N are plainly natural in N. □

Remark 3.6. Suppose that C is cocomplete. Then the adjoint $\overline{\tau}\colon C \to \overline{\mathrm{B}}A$ is a morphism of coalgebras and a chain equivalence. Indeed, the canonical comparison yields a chain inverse

$$\overline{\mathrm{B}}A \otimes_{\tau^{\mathrm{B}}} A \longrightarrow C \otimes_\tau A$$

which is in fact a morphism in the category of differential graded A-modules. This morphism descends to a chain map $\overline{\mathrm{B}}A \to C$, not necessarily compatible with the coalgebra structures but a chain homotopy inverse for $\overline{\tau}$. Consequently the induced map

$$\overline{\Omega}C \longrightarrow \overline{\Omega}\overline{\mathrm{B}}A \longrightarrow A$$

is a morphism of differential graded algebras and a chain equivalence whence the induced chain map

$$A^* \longrightarrow (\overline{\Omega}C)^* = \mathrm{Hom}(\overline{\Omega}C, R)$$

is a chain equivalence. Furthermore, since H^*C is supposed to be of finite type, the canonical map

$$\overline{\mathrm{B}}C^* \longrightarrow \mathrm{Hom}(\overline{\Omega}C, R)$$

is a chain equivalence. Since the homology of $\overline{\mathrm{B}}C^*$ is the differential torsion product $\mathrm{Tor}_{C^*}(R, R)$, we conclude that the homology of A^*, that is, the cohomology $\mathrm{H}^*(A)$ of A, is naturally isomorphic to $\mathrm{Tor}_{C^*}(R, R)$. Since $\mathrm{H}^*(C)$ is of finite type, so is $\mathrm{Tor}_{C^*}(R, R)$ whence $\mathrm{H}^*(A)$ is necessarily of finite type.

Remark 3.7. Suppose that C is itself of finite type, and let $M^\flat$ be a left C^*-module. By adjointness, the left C^*-module structure is then induced from a left C-comodule structure on $M^\flat$, and the twisted Hom-object $h^*(M^\flat)$ calculates the differential graded $\mathrm{Coext}_C(R, M^\flat)$.

Let N be a differential graded left A-module, and consider N^*, viewed as a differential graded right A-module. The twisted Hom-object

$$\mathrm{Hom}^\tau(C, N^*) \cong \mathrm{Hom}_A(C \otimes_\tau A, N^*)$$

acquires a canonical C^*-module structure and, furthermore, calculates the differential $\mathrm{Ext}_A(R, N^*)$. The bar construction $\mathrm{B}(R, C^*, \mathrm{Hom}^\tau(C, N^*))$ can be written as the twisted object

$$\mathrm{B}(R, C^*, \mathrm{Hom}^\tau(C, N^*)) = (\overline{\mathrm{B}}C^*) \otimes_{\tau^{\overline{\mathrm{B}}}} \mathrm{Hom}^\tau(C, N^*)$$

and the forgetful map $\mathrm{Hom}^\tau(C, N^*) \to N^*$ is a chain map and extends canonically to a projection

$$\pi_{N^*}\colon \mathrm{B}(R, C^*, \mathrm{Hom}^\tau(C, N^*)) \longrightarrow N^*. \tag{3.8}$$

For later reference, we will now explore the question under what circumstances this projection is a chain equivalence.

The projection (3.8) is formally similar to the projection ($3.5.1_*$) above. Furthermore, it is a kind of formal dual of a map

$$N \longrightarrow \overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N \tag{3.9}$$

of the kind ($3.5.6_*$) above. By Theorem 3.5_*, the map (3.5.4), in turn, fits into an $(\overline{\Omega}C)$-module contraction of the kind

$$(\overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N \underset{(3.4_*)}{\overset{(3.5.4)}{\leftrightarrows}} N, h_N) \tag{3.10}$$

where, relative to (3.4_*), we have substituted $\overline{\Omega}C$ for A. Dualizing this contraction, we arrive at the contraction

$$(\mathrm{Hom}(\overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N, R) \underset{(3.5.4)^*}{\overset{(3.4_*)^*}{\leftrightarrows}} N^*, h_N^*) \tag{3.11}$$

having the feature that $(3.4_*)^*$ is compatible with the additional structure. The object $\mathrm{Hom}(\overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N, R)$ can be written as the complete twisted Hom-object

$$\mathrm{Hom}^{\tau_{\overline{\Omega}}}(\overline{\Omega}C, \mathrm{Hom}^\tau(C, N^*))$$

Consider the canonical injection

$$\mathrm{B}(R, C^*, \mathrm{Hom}^\tau(C, N^*)) \longrightarrow \mathrm{Hom}(\overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N, R), \tag{3.12}$$

or, equivalently,

$$\overline{\mathrm{B}}C^* \otimes_{\tau^{\overline{\mathrm{B}}}} \mathrm{Hom}^\tau(C, N^*) \longrightarrow \mathrm{Hom}^{\tau_{\overline{\Omega}}}(\overline{\Omega}C, \mathrm{Hom}^\tau(C, N^*)). \tag{3.13}$$

The composite of (3.12) with the canonical forgetful map

$$\mathrm{Hom}(\overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N, R) \longrightarrow N^*$$

coincides with the projection (3.8).

Lemma 3.14. *Suppose that C is non-negative, simply connected and free as a graded R-module. When the homology $\mathrm{H}(C)$ of C is of finite type, the injection* (3.13) *is a chain equivalence whence the projection* (3.8) *is then a chain equivalence.*

Proof. We will exploit the construction in [44] (4.6): To adjust to the notation in the quoted reference, let $V = s^{-1}JC$ and $\mathrm{H} = \mathrm{H}(s^{-1}JC)$, the differential on $s^{-1}JC$ being that induced from the differential on C. The differential graded algebra $\overline{\Omega}C$ is one of the kind $(\mathrm{T}[V], d)$, the differential d being the ordinary coalgebra perturbation of the differential on $V = s^{-1}JC$. As in [44] (4.6), let $(F\mathrm{H}, \delta)$ be a free resolution of H which we take here to be of finite type. The construction in [44] (4.6) yields a

perturbed differential d' on the graded tensor algebra $\mathrm{T}[F\mathrm{H}]$ (perturbing the ordinary tensor algebra differential) and a filtered chain equivalence

$$(\mu', (\mathrm{T}[F\mathrm{H}], d') \underset{f}{\overset{g}{\leftrightarrows}} (\mathrm{T}[V], d), \nu') \tag{3.14.1}$$

of augmented differential graded algebras. By construction, the augmented differential graded algebra $\mathcal{A} = (\mathrm{T}[F\mathrm{H}], d')$ is of finite type and, since (3.14.1) is a filtered chain equivalence of augmented differential graded algebras, $f\colon \mathcal{A} \to \overline{\Omega}C$ and $g\colon \overline{\Omega}C \to A$ are morphisms of differential graded algebras which are also chain equivalences, indeed, the chain homotopies μ' and ν' are even compatible with the algebra structures.

Since $\mathcal{A}$ is of finite type, the dual $\mathcal{A}^*$ is a coaugmented differential graded coalgebra. Moreover, the composite

$$\tau_g\colon C \xrightarrow{\tau_{\overline{\Omega}}} \overline{\Omega}C \xrightarrow{g} \mathcal{A}$$

is a twisting cochain, and so is the dual $\tau_g^*\colon \mathcal{A}^* \to C^*$.

Instead of (3.13), we can now consider the canonical chain map

$$\mathcal{A}^* \otimes_{\tau_g^*} \mathrm{Hom}^\tau(C, N^*) \to \mathrm{Hom}^{\tau_g}(\mathcal{A}, \mathrm{Hom}^\tau(C, N^*)) \cong \mathrm{Hom}(\mathcal{A} \otimes_{\tau_g} C \otimes_\tau N, R)) \tag{3.14.2}$$

Since $\mathcal{A}$ is of finite type, this chain map is an isomorphism. The left-hand side of (3.14.2) is chain homotopic to $\overline{\mathrm{B}}C^* \otimes_{\tau^{\overline{\mathrm{B}}}} \mathrm{Hom}^\tau(C, N^*)$ compatibly with the bundle structure and the right-hand side of (3.14.2) is chain homotopic to $\mathrm{Hom}^{\tau_{\overline{\Omega}}}(\overline{\Omega}C, \mathrm{Hom}^\tau(C, N^*))$ compatibly with the bundle structures. Indeed, the adjoint $\overline{\tau}_g^*\colon \mathcal{A}^* \to \overline{\mathrm{B}}C^*$ of τ_g^* yields the chain map

$$\mathcal{A}^* \otimes_{\tau_g^*} \mathrm{Hom}^\tau(C, N^*) \longrightarrow \overline{\mathrm{B}}C^* \otimes_{\tau^{\overline{\mathrm{B}}}} \mathrm{Hom}^\tau(C, N^*),$$

necessarily compatible with the bundle structures and a chain equivalence. Likewise the morphism $g\colon \overline{\Omega}C \to \mathcal{A}$ of augmented differential graded algebras induces the morphism

$$g \otimes C \otimes N\colon \overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N \longrightarrow \mathcal{A} \otimes_{\tau_g} C \otimes_\tau N$$

of twisted objects, necessarily a chain equivalence. This morphism of bundles, in turn, dualizes to the chain equivalence

$$\mathrm{Hom}(\overline{\Omega}C \otimes_{\tau_{\overline{\Omega}}} C \otimes_\tau N, R) \longleftarrow \mathrm{Hom}(\mathcal{A} \otimes_{\tau_g} C \otimes_\tau N, R).$$

Putting the various items together we obtain the asserted chain equivalence. $\square$

4. Small models for ordinary equivariant (co)homology

Given the two simplicial sets S and T, the Eilenberg-Zilber theorem (cf. [16] II, Theorem 2.1) yields a contraction $(S \otimes T \underset{\nabla}{\overset{g}{\leftrightarrows}} S \times T, h)$; here we do not distinguish in notation between a simplicial set and its normalized chain complex. See e. g. [16] (I §5 and II §2), [25] (section 4, in particular p. 409), [28] (Proposition A.3) for details. The morphism ∇ is referred to as the *shuffle map* and g as the *Alexander-Whitney map*. In [16] (II, Theorem 2.1) the vanishing of the square hh of the homotopy h constructed there is not established but, in view of Remark 2.1.4 above, this

vanishing can always be achieved. The fact that the shuffle map is a morphism of differential graded coalgebras is given in [19] (§17), cf. also Lemma 4.4 in [28]. A somewhat more general form of the Eilenberg-Zilber theorem, due to Cartier, can be found in [14] (2.9). Below we will apply the Eilenberg-Zilber theorem to the singular set SY of a topological space Y and to the simplicial bar construction. The Eilenberg-Zilber theorem is well known to imply that the normalized chain complex C_*S of a simplicial set S acquires a differential graded coalgebra structure; moreover, a choice of base point determines a coaugmentation for C_*S, and C_*S has a unique coaugmentation and is cocomplete if and only if S is *reduced* in the sense that S_0 consists of a single point. Now, given a pathwise and locally pathwise connected topological space Y, the injection of the first Eilenberg subcomplex S^1Y relative to a chosen base point o (which consists of all singular simplices mapping each vertex to the base point o) into the singular set SY associated to Y is an injection of differential graded coalgebras and a chain equivalence; thus we can then take $C_*(Y)$ to be a cocomplete differential graded coalgebra.

Let G be a topological group. Before we proceed further we note the following. Suppose that the group G is connected. Then we can replace the singular set SG associated to G—this is a simplicial group—with the first Eilenberg subcomplex S^1G relative to the neutral element of G; the first Eilenberg subcomplex S^1G is a simplicial subgroup and the injection of S^1G into SG is a morphism of simplicial groups which is a chain equivalence. The normalized chain algebra of S^1G is connected. Thus, given a connected topological group G, we can take C_*G to be this normalized chain algebra.

Let now G be a general topological group, X a left G-space and E a right G-space. It is well known that the Eilenberg-Zilber theorem has the following consequences: (i) *The chain complex C_*G of normalized singular chains on G acquires a differential graded Hopf algebra structure*; (ii) *this Hopf algebra C_*G acts from the left on the normalized singular chains $C_*(X)$ and from the right on the normalized singular cochains $C^*(X)$ on X.*

The construction of the small models for ordinary equivariant (co)homology relies on the following folk-lore result, cf. [28] (Theorem 3.9), the significance of which was already commented on in the introduction.

Theorem 4.1. *The G-equivariant homology $\mathrm{H}^G_*(X,R)$ of X is canonically isomorphic to the differential $\mathrm{Tor}^{C_*G}_*(R,C_*X)$ and, likewise, the G-equivariant cohomology $\mathrm{H}^*_G(X,R)$ of X is canonically isomorphic to the differential $\mathrm{Ext}^*_{C_*G}(R,C^*X)$. The isomorphisms are natural in the data.*

Consider the $(G\times G)$-action on X which is the given action on the first copy of G and the trivial action on the second one. The diagonal map $G\to G\times G$ induces the morphisms

$$\mathrm{H}^G_*(X,R)\longrightarrow \mathrm{H}^{G\times G}_*(X,R) \tag{4.1.1}$$

$$\mathrm{H}^*_G(X,R)\otimes \mathrm{H}^*(BG,R)\longrightarrow \mathrm{H}^*_G(X,R) \tag{4.1.2}$$

of graded R-modules that satisfy the standard associativity constraints so that, in particular, $\mathrm{H}^*_G(X,R)$ acquires a graded $\mathrm{H}^*(BG,R)$-module structure and that, when $\mathrm{H}_*(BG,R)$ is projective as a graded R-module, $\mathrm{H}_*(BG,R)$ acquires a graded

cocommutative coalgebra structure and $\mathrm{H}^*_G(X,R)$ a graded comodule structure over $\mathrm{H}_*(BG,R)$. Likewise, the diagonal map of G induces the morphisms

$$\mathrm{Tor}^{C_*G}_*(R,C_*X) \longrightarrow \mathrm{Tor}^{C_*(G\times G)}_*(R,C_*X) \tag{4.1.3}$$

$$\mathrm{Ext}^*_{C_*G}(R,C^*X)\otimes \mathrm{Ext}^*_{C_*G}(R,R) \longrightarrow \mathrm{Ext}^*_{C_*G}(R,C^*X) \tag{4.1.4}$$

of graded R-modules that satisfy the standard associativity constraints so that, in particular, $\mathrm{Ext}^*_{C_*G}(R,R)\cong \mathrm{H}^*(BG)$ is a graded commutative algebra and that $\mathrm{Ext}^*_{C_*G}(R,C^*X)$ acquires a graded $\mathrm{Ext}^*_{C_*G}(R,R)$-module structure via (4.1.4). Moreover, when $\mathrm{Tor}^{C_*G}_*(R,R)\cong \mathrm{H}_*(BG)$ is projective as a graded R-module, $\mathrm{Tor}^{C_*G}_*(R,R)$ acquires a graded cocommutative coalgebra structure and $\mathrm{Tor}^{C_*G}_*(R,C_*X)$ acquires a graded comodule structure over the graded coalgebra $\mathrm{Tor}^{C_*G}_*(R,R)$ via (4.1.3).

Addendum 1. *The naturality of the constructions implies that the isomorphisms between $\mathrm{H}^G_*(X,R)$ and $\mathrm{Tor}^{C_*G}_*(R,C_*X)$ and between $\mathrm{H}^*_G(X,R)$ and $\mathrm{Ext}^*_{C_*G}(R,C^*X)$ are compatible with the additional structure just explained.*

Addendum 2. *More generally, when E is a principal G-space, there are canonical isomorphims of the kind*

$$\mathrm{H}_*(E\times_G X,R)\cong \mathrm{Tor}^{C_*G}_*(C_*E,C_*X),\quad \mathrm{H}^*(E\times_G X,R)\cong \mathrm{Ext}^*_{C_*G}(C_*E,C^*X)$$

which are natural in the data.

We now explain briefly how Theorem 4.1 is established by means of certain HPT-techniques needed later in the paper anyway. To this end, we denote by $\tau_G\colon \overline{\mathrm{B}}C_*G\to C_*G$ the universal twisting cochain (which is acyclic) for the (reduced normalized) bar construction $\overline{\mathrm{B}}C_*G$ of the augmented differential graded algebra C_*G. As explained earlier, by taking the first Eilenberg subcomplex, we will henceforth take $C_*(BG)$ to be connected, so that $C_*(BG)$ acquires a cocomplete coaugmented differential graded coalgebra structure. We remind the reader that $\overline{\mathrm{B}}C_*G$ is automatically cocomplete.

Lemma 4.2. *The data determine an acyclic twisting cochain*

$$\tau\colon C_*(BG)\to C_*G. \tag{4.2.1}$$

Furthermore, the data determine a chain map from $C_*(EG\times_G X)$ to $(\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*X$ which is a chain equivalence and a morphism of differential graded $(\overline{\mathrm{B}}C_*G)$-comodules via the adjoint $\overline{\tau}\colon C_*(BG)\to \overline{\mathrm{B}}C_*G$ of τ. This chain map is natural in X, G, and the G-action.

Proof. The twisted *Eilenberg-Zilber* theorem [7], [25] yields a twisting cochain $\tau\colon C_*BG\to C_*G$ and a contraction

$$\left((C_*BG)\otimes_\tau C_*X \underset{\nabla}{\overset{\pi}{\leftrightarrows}} C_*(EG\times_G X), h\right) \tag{4.2.2}$$

of $C_*(EG\times_G X)$ onto $(C_*BG)\otimes_\tau C_*X$ in such a way that π is a morphism of (C_*BG)-comodules. This contraction is, furthermore, natural in the data. The

adjoint $\overline{\tau}\colon C_*BG \to \overline{\mathrm{B}}C_*G$ of τ is a morphism of differential graded coalgebras as well as a chain equivalence; it induces, in turn, the chain map

$$\overline{\tau}\otimes \mathrm{Id}\colon (C_*BG)\otimes_\tau C_*X \longrightarrow (\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*X \tag{4.2.3}$$

which, via $\overline{\tau}$, is a morphism of differential graded $(\overline{\mathrm{B}}C_*G)$-comodules. A standard spectral sequence comparison argument shows that (4.2.3) is a chain equivalence. Indeed, the filtrations of C_*BG and $\overline{\mathrm{B}}C_*G$ relative to the ordinary degree yield Serre spectral sequences and (4.2.3) induces a morphism of spectral sequences. Both spectral sequences have E_2 isomorphic to $\mathrm{H}_*(BG, \mathrm{H}_*(X))$ and (4.2.3) induces an isomorphism of spectral sequences from E_2 on whence (4.2.3) is an isomorphism on homology. Since the chain complexes on both sides of (4.2.3) are free over the ground ring, (4.2.3) is a chain equivalence.

The following reasoning avoids spectral sequences: The twisted objects $(C_*BG)\otimes_\tau C_*G$ and $(\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*G$ are both acyclic constructions for the differential graded algebra C_*G, cf. e. g. [49] for the notion of acyclic construction, and choices of contracting homotopies for these constructions determine canonical (C_*G)-linear comparison maps

$$\alpha\colon (C_*BG)\otimes_\tau C_*G \longrightarrow (\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*G, \quad \beta\colon (\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*G \longrightarrow (C_*BG)\otimes_\tau C_*G$$

together with chain homotopies $\alpha\beta \cong \mathrm{Id}$ and $\beta\alpha \cong \mathrm{Id}$, cf. e. g. [49] for details. Things may in fact be arranged in such a way that

$$\alpha = \overline{\tau}\otimes \mathrm{Id}\colon (C_*BG)\otimes_\tau C_*G \longrightarrow (\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*G.$$

Now β induces a morphism

$$\beta^\sharp\colon (\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*X \longrightarrow (C_*BG)\otimes_\tau C_*X,$$

and the two chain homotopies induce the requisite chain homotopies so that $\beta^\sharp$ and (4.2.3) are chain homotopy inverse to each other. Notice that the chain inverse $\beta^\sharp$ need not be compatible with the $(\overline{\mathrm{B}}C_*G)$-comodule structures, though.

The two chain equivalences π and (4.2.3) combine to the chain equivalence

$$C_*(EG\times_G X) \xrightarrow{\pi} (C_*BG)\otimes_\tau C_*X \xrightarrow{\overline{\tau}\otimes\mathrm{Id}} (\overline{\mathrm{B}}C_*G)\otimes_{\tau_G} C_*X$$

which, by construction, is a morphism of differential graded $(\overline{\mathrm{B}}C_*G)$-comodules via the adjoint $\overline{\tau}\colon C_*(BG) \to \overline{\mathrm{B}}C_*G$ of τ. The construction is natural in X, G, and the G-action. □

We note that the proof of the twisted Eilenberg-Zilber theorem in [25] involves the *perturbation lemma*. Thus Lemma 4.2 relies on HPT as well.

Let $\xi\colon P \to B$ be a principal right G-bundle, let F be a left G-space, and consider the associated fiber bundle $P\times_G F \to B$. Let C be a differential graded coalgebra, let $\phi\colon C_*B \to C$ be a morphism of differential graded coalgebras which is as well a chain equivalence, and let $\vartheta\colon C \to C_*G$ be a twisting cochain. The twisted tensor product $C\otimes_\vartheta C_*F$ together with a chain map

$$C_*(P\times_G F) \longrightarrow C\otimes_\vartheta C_*F$$

which is a chain equivalence and a morphism of C-comodules via ϕ is a model for the chains of $P \times_G F$ that is, furthermore, compatible with the induced C-comodule structures. Likewise, the twisted Hom-object $\mathrm{Hom}^\vartheta(C, C_*F)$ together with a (co)chain map

$$\mathrm{Hom}^\vartheta(C, C_*F) \to C^*(P \times_G F)$$

which is a chain equivalence and a morphism of C^*-modules via $\phi^*\colon C^* \to C^*B$ is a model for the cochains of $P \times_G F$ that is, furthermore, compatible with the induced C^*-module structures.

We now spell out a consequence of Lemma 4.2 which, in turn, immediately implies the statement of Theorem 4.1.

Corollary 4.3. *The twisted tensor product* $(\overline{\mathrm{B}}C_*G) \otimes_{\tau_G} C_*X$, *together with* (4.2.3) *and either of the morphisms* π *or* ∇ *in* (4.2.2), *is a model for the chains of the Borel construction* $EG \times_G X$. *Likewise the twisted* Hom-*object* $\mathrm{Hom}^{\tau_G}(\overline{\mathrm{B}}C_*(G), C^*X)$, *together with the corresponding isomorphism of the kind* (1.2.6) *and the duals of the comparison maps between* $(\overline{\mathrm{B}}C_*G) \otimes_{\tau_G} C_*X$ *and* $C_*(EG \times_G X)$ *just spelled out is a model for the cochains of the Borel construction* $EG \times_G X$. *These models are natural in* X, G, *and the* G-*action.* $\square$

A variant of Theorem 4.1 merely for cohomology is obtained in the following manner: Since in general the dual of a tensor product is not the tensor product of the duals, the cochains C^*G on G do not inherit a coalgebra structure; a replacement for the missing comodule structure on C^*X (dual to the C_*G-module structure on C_*X exploited above) is provided by the following construction: The dual $\overline{\mathrm{B}}^*(C_*G)$ of the reduced bar construction $\overline{\mathrm{B}}(C_*G)$ acquires a differential graded algebra structure. The cochains $C^*(EG \times_G X)$ inherit a canonical $C^*(BG)$-module structure and hence, via the dual of the adjoint of the twisting cochain (4.2.1), a canonical $\overline{\mathrm{B}}^*(C_*G)$-module structure. The forgetful map from $C^*(EG \times_G X)$ to C^*X (which forgets the G-action) is a chain map which extends canonically to a projection

$$\pi\colon \mathrm{B}(R, \overline{\mathrm{B}}^*(C_*G), C^*(EG \times_G X)) \to C^*(X). \tag{4.1.1*}$$

When G is connected and when the homology $\mathrm{H}_*(BG)$ of BG is of finite type, in view of Lemma 3.14, π is an isomorphism on homology. We will refer to $\mathrm{B}(R, \overline{\mathrm{B}}^*(C_*G), C^*(EG \times_G X))$ as the *extended cochain complex* of X, with reference to G. For intelligibility, we recall that, by the construction of the extended cochain complex, this complex inherits a canonical (differential graded) $\overline{\mathrm{B}}\,\overline{\mathrm{B}}^*(C_*G)$-comodule structure. We note that, for the present constructions, we cannot naively work with a reduced cobar construction on C^*G since the latter differential graded algebra does not acquire a coalgebra structure.

The following is a variant of Theorem 4.1, in the realm of ordinary singular *co*homology.

Theorem 4.1*. *Suppose that* G *is connected and that the homology* $\mathrm{H}_*(BG)$ *is of finite type. Then the differential graded coalgebra* $\overline{\mathrm{B}}\,\overline{\mathrm{B}}^*(C_*G)$ *is a model for the cochains on* G *in such a way that the diagonal map of* $\overline{\mathrm{B}}\,\overline{\mathrm{B}}^*(C_*G)$ *serves as a replacement for the missing coalgebra structure on* C^*G *and that the* G-*equivariant cohomology of* X *is canonically isomorphic to the differential graded*

$$\mathrm{Cotor}^{\overline{\mathrm{B}}\,\overline{\mathrm{B}}^*(C_*G)}(R, \mathrm{B}(R, \overline{\mathrm{B}}^*(C_*G), C^*(EG \times_G X))). \tag{4.1.2*}$$

Proof. With the notation $A = \overline{\mathrm{B}}^*(C_*G)$ and $\tau^{\overline{\mathrm{B}}}\colon \overline{\mathrm{B}}A \to A$ for the universal twisting cochain for the (reduced normalized) bar construction on A, the differential graded Cotor is the homology of the twisted object $A \otimes_{\tau^{\overline{\mathrm{B}}}} \mathrm{B}(R, A, C^*(EG \times_G X))$ which can be written as

$$A \otimes_{\tau^{\overline{\mathrm{B}}}} \overline{\mathrm{B}}A \otimes_{\tau^{\overline{\mathrm{B}}}} C^*(EG \times_G X),$$

and the morphism (3.4_*) yields a projection onto $C^*(EG \times_G X)$ which is in fact a relatively projective resolution of $C^*(EG \times_G X)$ in the category of (left) $\overline{\mathrm{B}}^*(C_*G)$-modules. □

We now proceed towards the construction of small models for equivariant (co)homology. The group G is said to be of *strictly exterior type* (over the ground ring R) provided, as a graded Hopf algebra, its homology $\mathrm{H}_*(G)$ is the exterior algebra $\Lambda[x_1, \ldots]$ in odd degree primitive generators $x_1, \ldots$; when the number of generators is infinite, we will assume throughout that $\mathrm{H}_*(G)$ is of finite type. With this assumption, in each degree, $\mathrm{H}_*(G)$ is plainly a free R-module of finite dimension. A detailed discussion of the property of being of strictly exterior type can be found in [46] (IV.7). When G is of strictly exterior type, it is necessarily connected, and $\mathrm{S}' = \mathrm{H}_*(BG)$ is a *simply connected* graded cocommutative coalgebra, indeed, the graded symmetric coalgebra on the suspension of the free R-module generated by $x_1, \ldots$ (the module of indecomposables). Henceforth we suppose that G is of strictly exterior type in such a way that the duals of the primitive generators are universally transgressive. The group G being of strictly exterior type, it is necessarily connected and, in view of an observation made earlier, we can take C_*G to be a connected differential graded algebra if need be.

Recall for intelligibility that, given the fibration $f\colon E \to B$, with typical fiber F, the differential $d_p\colon \mathrm{E}_p^{0,p} \longrightarrow \mathrm{E}_p^{p+1,0}$ $(p \geq 2)$ in the cohomology spectral sequence (E_*, d_*) of the fibration determines an additive relation

$$\tau\colon \mathrm{E}_2^{0,p} \rightharpoonup \mathrm{E}_2^{p+1,0}$$

referred to as *transgression*, cf. [48] (XI.4 p. 332); this is not the original definition of transgression but it is an equivalent notion. The elements of $\mathrm{E}_p^{0,p} \subseteq \mathrm{E}_2^{0,p} \cong \mathrm{H}^p(F)$ are then referred to as *transgressive*. When f is the universal bundle of a Lie group, the transgressive elements are said to be *universally transgressive*.

The following lemma yields an sh-morphism from C_*G to H_*G which is a quasi-isomorphism.

Lemma 4.4$_*$. *For each x_j, choose a cocycle of $\overline{\mathrm{B}}C_*(G)$ such that the dual of x_j in H^*G transgresses to the class of this cocycle. This choice determines an acyclic twisting cochain*

$$\zeta^B\colon \overline{\mathrm{B}}C_*G \to \mathrm{H}_*G. \qquad (4.4.1_*)$$

Consequently, for any differential graded right $(\overline{\mathrm{B}}C_*G)$-comodule M, the morphism (3.3_*) *(for right comodules rather than left ones) yields an injection*

$$M \longrightarrow M \otimes_{\zeta^B} (\mathrm{H}_*G) \otimes_{\zeta^B} (\overline{\mathrm{B}}C_*G) \qquad (4.4.2_*)$$

of M into the relatively injective twisted object $M \otimes_{\zeta^B} (\mathrm{H}_*G) \otimes_{\zeta^B} (\overline{\mathrm{B}}C_*G)$ which is a morphism of right $(\overline{\mathrm{B}}C_*G)$-comodules and a chain equivalence and hence serves as a

replacement for a relatively injective resolution of M in the category of differential graded right $(\overline{\mathrm{B}}C_*G)$-comodules.

A version of this lemma may be found in Section 4 of [30]. Alternative constructions for a twisting cochain of the kind $(4.4.1_*)$ may be found in [46] (IV.7.3) or may be deduced from a corresponding construction in [28]. Since [30] is not easily available, and for later reference, we sketch a proof of the lemma.

Proof. The Eilenberg-Zilber theorem yields a contraction

$$(\overline{\mathrm{B}}C_*G \otimes \overline{\mathrm{B}}C_*G \underset{\nabla}{\overset{\pi}{\leftrightarrows}} \overline{\mathrm{B}}(C_*G \otimes C_*G), h), \tag{4.4.3}$$

and, as noted earlier, the shuffle map ∇ is well known to be a morphism of differential graded coalgebras. Write $\Lambda = \mathrm{H}_*(G) = \Lambda[x_1, \dots]$ and, for each $j \geq 1$, write $\Lambda^j = \Lambda[x_1, \dots, x_j]$. For each x_j, the chosen cocycle amounts to a twisting cochain $\zeta_j \colon \overline{\mathrm{B}}C_*(G) \to \Lambda[x_j]$. By induction, we construct a sequence of twisting cochains $\zeta^j \colon \overline{\mathrm{B}}C_*(G) \to \Lambda^j$ $(j \geq 1)$.

The induction starts with $\zeta^1 = \zeta_1$. Let $j \geq 1$, and suppose, by induction, that the twisting cochain $\zeta^j \colon \overline{\mathrm{B}}C_*(G) \to \Lambda^j$ has been constructed. Then

$$\zeta^j \otimes \eta\varepsilon + \eta\varepsilon \otimes \zeta_{j+1} \colon \overline{\mathrm{B}}C_*(G) \otimes \overline{\mathrm{B}}C_*(G) \longrightarrow \Lambda^j \otimes \Lambda[x_{j+1}] = \Lambda^{j+1}$$

is a twisting cochain. The construction $(2.2.1_*)$, applied to $\sigma = \zeta^j \otimes \eta\varepsilon + \eta\varepsilon \otimes \zeta_{j+1}$ and (4.4.3), yields a twisting cochain from $\overline{\mathrm{B}}(C_*G \otimes C_*G)$ to Λ^{j+1} which, combined with

$$\overline{\mathrm{B}}\Delta \colon \overline{\mathrm{B}}C_*G \to \overline{\mathrm{B}}(C_*G \otimes C_*G),$$

yields the twisting cochain $\zeta^{j+1} \colon \overline{\mathrm{B}}C_*G \longrightarrow \Lambda^{j+1}$. This completes the inductive step. When the generators of H_*G constitute a finite set, the construction stops after finitely many steps and yields the twisting cochain ζ^B; otherwise, in view of the assumption that H_*G be of finite type, the ζ^j's converge to a twisting cochain ζ^B; in fact, the convergence is naive in the sense that, in each degree, the limit is achieved after finitely may steps.

An elementary spectral sequence argument shows that the construction $(\overline{\mathrm{B}}C_*G) \otimes_{\zeta^B} \mathrm{H}_*G$ is acyclic. Hence the twisting cochain ζ^B is acyclic. $\square$

Under the circumstances of Lemma 4.4, extend the adjoint $\overline{\zeta}^B \colon \overline{\Omega}\,\overline{\mathrm{B}}C_*(G) \to \mathrm{H}_*G$ of ζ^B to a contraction

$$(\mathrm{H}_*G \underset{\nabla}{\overset{\overline{\zeta}^B}{\leftrightarrows}} \overline{\Omega}\,\overline{\mathrm{B}}C_*(G), h) \tag{4.5.1}$$

where the notation ∇ and h is abused somewhat. Indeed, by construction, $\overline{\zeta}^B$ is surjective and an isomorphism on homology and $\overline{\Omega}\,\overline{\mathrm{B}}C_*(G)$ and H_*G are both free as graded modules over the ground ring. Consequently the kernel of $\overline{\zeta}^B$ has zero homology and is even contractible, and we can extend $\overline{\zeta}^B$ to data of the kind (2.1.1) satisfying (2.1.2). As explained in Remark 2.1.4, a suitable modification of the homotopy if need be then yields a homotopy such that the side conditions (2.1.3) are satisfied as well.

The following lemma yields an sh-inverse of the sh-morphism from C_*G to H_*G constructed in Lemma 4.4_* above.

Lemma 4.5. *Under the circumstances of Lemma 4.4, the choice of cocycle of* $\overline{\mathrm{B}}C_*(G)$ *for each* x_j *and the contraction* (4.5.1) *determine an acyclic twisting cochain*

$$\vartheta\colon \overline{\mathrm{B}}\mathrm{H}_*(G) \to C_*G \tag{4.5.2}$$

such that the adjoints $\overline{\mathrm{B}}C_*G \to \overline{\mathrm{B}}\mathrm{H}_*(G)$ *and* $\overline{\mathrm{B}}\mathrm{H}_*(G) \to \overline{\mathrm{B}}C_*G$ *of* (4.4.1$_*$) *and* (4.5.2), *respectively, are mutually chain homotopy inverse to each other. In particular, the composite*

$$\mathrm{H}_*(BG) \to \overline{\mathrm{B}}C_*G \tag{4.5.3}$$

of the canonical injection $\mathrm{H}_*(BG) \to \overline{\mathrm{B}}\mathrm{H}_*(G)$ *with the adjoint* $\overline{\mathrm{B}}\mathrm{H}_*(G) \to \overline{\mathrm{B}}C_*G$ *of* (4.5.2) *is a morphism of differential graded coalgebras inducing an isomorphism on homology.*

Proof. The construction (2.2.1*), applied to $\sigma = \tau_{\overline{\mathrm{B}}\mathrm{H}_*G}\colon \overline{\mathrm{B}}\mathrm{H}_*G \to \mathrm{H}_*G$ (the universal twisting cochain for H_*G) and (4.5.1), yields a twisting cochain $\overline{\mathrm{B}}\mathrm{H}_*G \to \overline{\Omega}\,\overline{\mathrm{B}}C_*(G)$. The composite of this twisting cochain with the universal morphism $\overline{\Omega}\,\overline{\mathrm{B}}C_*(G) \to C_*(G)$ of differential graded algebras yields the twisting cochain (4.5.2).

A spectral sequence argument shows that the construction $\overline{\mathrm{B}}\mathrm{H}_*(G) \otimes_\vartheta \mathrm{C}_*G$ is acyclic. Hence the twisting cochain (4.5.2) is acyclic.

Standard comparison arguments involving the two acyclic constructions $(\overline{\mathrm{B}}C_*G) \otimes_{\zeta^B} \mathrm{H}_*G$ and $(\overline{\mathrm{B}}\mathrm{H}_*(G)) \otimes_\vartheta \mathrm{C}_*G$ show that the adjoints of (4.4.1$_*$) and (4.5.2) are mutually chain homotopy inverse to each other. □

We will denote the twisting cochain which is the composite of (4.5.3) with the universal twisting cochain τ_G of C_*G by

$$\zeta_G\colon \mathrm{H}_*(BG) \to C_*G. \tag{4.4.1*}$$

This twisting cochain is plainly acyclic. It is logically the adjoint (or dual) of the twisting cochain (4.4.1$_*$) whence the numbering. The description of (4.4.1*) involves the injection (4.5.3), though, whence there seems no way to avoid this kind of numbering.

The statement of the following lemma is now immediate; yet we spell it out to bring the duality between G and BG to the fore.

Lemma 4.4*. *For any differential graded left* (C_*G)*-module* N*, the morphism* (3.4$_*$) *yields a* (C_*G)*-linear projection*

$$(C_*G) \otimes_{\zeta_G} (\mathrm{H}_*BG) \otimes_{\zeta_G} N \to N \tag{4.4.2*}$$

from the relatively projective twisted object $(C_*G) \otimes_{\zeta_G} (\mathrm{H}_*BG) \otimes_{\zeta_G} N$ *onto* N *which is a chain equivalence and hence serves as a replacement for a relatively projective resolution of* N *in the category of differential graded left* (C_*G)*-modules.* □

Theorem 4.6. *The morphism*

$$\overline{\zeta_G} \otimes \mathrm{Id}\colon \mathrm{H}_*(BG) \otimes_{\zeta_G} C_*X \to (\overline{\mathrm{B}}C_*(G)) \otimes_{\tau_G} C_*X \tag{4.6.1$_*$}$$

of twisted tensor products induces an isomorphim on homology, and so does the morphism

$$\mathrm{Hom}^{\zeta_G}(\mathrm{H}_*(BG), C^*X) \leftarrow \mathrm{Hom}^{\tau_G}(\overline{\mathrm{B}}C_*(G), C^*X). \tag{4.6.1*}$$

Consequently the twisted tensor product

$$\mathrm{H}_*(BG) \otimes_{\zeta_G} C_*X \tag{4.6.2$_*$}$$

is a model for chains of the Borel construction of X in the sense that it calculates the G-equivariant homology of X as an $\mathrm{H}_*(BG)$-comodule, and the twisted Hom-object

$$\mathrm{Hom}^{\zeta_G}(\mathrm{H}_*(BG), C^*X) \cong \mathrm{H}^*(BG) \otimes_{\zeta_G^*} C^*X \tag{4.6.2*}$$

is a model for cochains of the Borel construction of X in the sense that it calculates the G-equivariant cohomology of X as an $\mathrm{H}^*(BG)$-module.

Proof. Standard spectral sequence comparison arguments establish the first claim. The following reasoning avoids spectral sequences.

Extend the adjoint $\overline{\zeta_G}\colon \mathrm{H}_*(BG) \to \overline{\mathrm{B}}C_*G$ of the acyclic twisting cochain (4.4.1*) to a contraction

$$(\mathrm{H}_*BG \underset{\overline{\zeta_G}}{\overset{\pi}{\leftrightarrows}} \overline{\mathrm{B}}C_*(G), h). \tag{4.6.3}$$

This contraction, in turn, induces the contraction

$$((\mathrm{H}_*BG) \otimes C_*X \underset{\overline{\zeta_G}\otimes \mathrm{Id}}{\overset{\pi\otimes\mathrm{Id}}{\leftrightarrows}} (\overline{\mathrm{B}}C_*(G)) \otimes C_*X, h \otimes \mathrm{Id}). \tag{4.6.4}$$

The twisted differentials on both sides of (4.6.1$_*$) are perturbed differentials, and an application of the perturbation lemma yields a contraction of the kind

$$((\mathrm{H}_*BG) \otimes_{\zeta_G} C_*X \underset{\overline{\zeta_G}\otimes \mathrm{Id}}{\overset{\widetilde{\pi}}{\leftrightarrows}} (\overline{\mathrm{B}}C_*(G)) \otimes_{\tau_G} C_*X, \widetilde{h}). \tag{4.6.5}$$

Hence (4.6.1$_*$) is in particular a chain equivalence. Passing to the appropriate duals, we see that (4.6.1*) is likewise a chain equivalence. □

Remark. Theorem 4.6 is implicit in [30] but this reference is not generally available. At the time when [30] was written, there was little interest in this kind of result.

5_*. Koszul duality for homology

Let V be a free graded R-module of finite type concentrated in odd positive degrees, let $\Lambda = \Lambda[V]$, the exterior R-algebra on V, and let $\mathrm{S}' = \mathrm{S}'[sV]$, the symmetric coalgebra on the suspension sV; thus Λ is a strictly exterior R-algebra of finite type, S' is the symmetric coalgebra on the suspension sV of the indecomposables of Λ, and the canonical injection $\mathrm{S}' \to \overline{\mathrm{B}}\Lambda$ is a homology isomorphism. Notice that both S' and $\overline{\mathrm{B}}\Lambda$ are simply connected. To adjust to the notation established in Subsection 1.6 and in Section 3 above, let

$$(C, A, C', A', \zeta^{C'}, \zeta_{A'}) = (\mathrm{S}', \Lambda, \overline{\mathrm{B}}\Lambda, \overline{\Omega}\mathrm{S}', \tau^{\overline{\mathrm{B}}\Lambda}, \tau_{\overline{\Omega}\mathrm{S}'})$$

and let $\tau\colon \mathrm{S}' \to \Lambda$ be the universal (acyclic) twisting cochain determined by the desuspension map $sV \to V$. For simplicity, write $\tau_{\overline{\Omega}} = \tau_{\overline{\Omega}\mathrm{S}'}$ and $\tau^{\overline{\mathrm{B}}} = \tau^{\overline{\mathrm{B}}\Lambda}$.

Non-negatively graded objects will be indicated with the notation ${}_{\geq 0}$ if need be. Consider the functors

$$(5.1_*) \qquad t^\infty\colon {}_\Lambda\mathrm{Mod}^\infty \longrightarrow {}_{\mathrm{S}'}\mathrm{Comod}, \quad t^\infty(N, \tau_{\overline{\Omega}}) = \mathrm{S}' \otimes_{\tau_{\overline{\Omega}}} N$$

$$(5.2_*) \qquad h^\infty\colon {}_{\mathrm{S}'}\mathrm{Comod}^\infty \longrightarrow {}_\Lambda\mathrm{Mod}, \quad h^\infty(M, \tau^{\overline{\mathrm{B}}}) = \Lambda \otimes_{\tau^{\overline{\mathrm{B}}}} M.$$

Plainly, these functors restrict to functors

$$t^\infty\colon {}_\Lambda\mathrm{Mod}^\infty_{\geq 0} \longrightarrow {}_{\mathrm{S}'}\mathrm{Comod}_{\geq 0}, \quad h^\infty\colon {}_{\mathrm{S}'}\mathrm{Comod}^\infty_{\geq 0} \longrightarrow {}_\Lambda\mathrm{Mod}_{\geq 0}.$$

Proposition 5.3_*. *The functors t^∞ and h^∞ are quasi-inverse to each other and yield a quasi-equivalence of categories between ${}_\Lambda\mathrm{Mod}^\infty$ and ${}_{\mathrm{S}'}\mathrm{Comod}^\infty$ which restricts to a quasi-equivalence of categories between ${}_\Lambda\mathrm{Mod}^\infty_{\geq 0}$ and ${}_{\mathrm{S}'}\mathrm{Comod}^\infty_{\geq 0}$.*

We now explain the meaning of this proposition and in particular that of the terms "quasi-inverse" and "quasi-equivalence": For any left sh-comodule $(M, \tau^{\overline{\mathrm{B}}})$ over S', the relatively injective object $(3.3)_*$ with $C = \overline{\mathrm{B}}\Lambda$ has the form

$$M \longrightarrow \overline{\mathrm{B}}\Lambda \otimes_{\tau^{\overline{\mathrm{B}}}} \Lambda \otimes_{\tau^{\overline{\mathrm{B}}}} M;$$

this injection and the obvious injection

$$t(h^\infty(M, \tau^{\overline{\mathrm{B}}})) = \mathrm{S}' \otimes_\tau \Lambda \otimes_{\tau^{\overline{\mathrm{B}}}} M \longrightarrow \overline{\mathrm{B}}\Lambda \otimes_{\tau^{\overline{\mathrm{B}}}} \Lambda \otimes_{\tau^{\overline{\mathrm{B}}}} M$$

are both morphisms of left $\overline{\mathrm{B}}\Lambda$-comodules, that is, morphisms of sh-comodules over S', both morphisms are quasi-isomorphisms, and these are natural in the sh-comodule $(M, \tau^{\overline{\mathrm{B}}})$ over S'. Likewise, for any left sh-module $(N, \tau_{\overline{\Omega}})$ over Λ, the relatively projective object $(3.4)_*$ with $A = \overline{\Omega}\mathrm{S}'$ has the form

$$\overline{\Omega}\mathrm{S}' \otimes_{\tau_{\overline{\Omega}}} \mathrm{S}' \otimes_{\tau_{\overline{\Omega}}} N \longrightarrow N;$$

this map and the obvious surjection

$$\overline{\Omega}\mathrm{S}' \otimes_{\tau_{\overline{\Omega}}} \mathrm{S}' \otimes_{\tau_{\overline{\Omega}}} N \longrightarrow \Lambda \otimes_\tau \mathrm{S}' \otimes_{\tau_{\overline{\Omega}}} N = h(t^\infty(N, \tau_{\overline{\Omega}}))$$

are both morphisms of left $(\overline{\Omega}\mathrm{S}')$-modules, that is, morphisms of sh-modules over Λ, both morphisms are quasi-isomorphisms, and these are natural in the sh-module $(N,\tau_{\overline{\Omega}})$ over Λ. It is in this sense that the functors t^∞ and h^∞ are *quasi-equivalences which are quasi-inverse to each other*. These observations imply a proof of Proposition 5.3_*.

5.4_*. Koszul duality in ordinary equivariant homology, cf. [21], [22], [24]. Let G be a topological group of strictly exterior type such that the duals of the exterior generators are universally transgressive and let $\Lambda = \mathrm{H}_*G$ and $\mathrm{S}' = \mathrm{H}_*(BG)$. *Any left (C_*G)-module is a left sh-module over H_*G via the twisting cochain* $\zeta_G\colon \mathrm{H}_*(BG) \to C_*G$ given in $(4.4.1^*)$ and *any left $(\overline{\mathrm{B}}C_*G)$-comodule is a left sh-comodule over* $\mathrm{H}_*(BG)$ via the twisting cochain $\zeta^B\colon \overline{\mathrm{B}}C_*G \to \mathrm{H}_*G$ given in $(4.4.1_*)$. In particular, given a left G-space X, the chain complex C_*X is a left sh-module over H_*G via ζ_G and, for any space Y over BG, the chains C_*Y constitute a left sh-comodule over $\mathrm{H}_*(BG)$ via ζ^B and the twisting cochain (4.2.1). Thus the functors (5.1_*) and (5.2_*) are defined, and the *statement of Proposition 5.3_* entails that ordinary and equivariant homology are related by Koszul duality in the following sense: On the category of (left) G-spaces, the functor $h \circ t^\infty \circ C_*$ is chain-equivalent to the functor C_* as sh-module functors over $\Lambda = \mathrm{H}_*G$; and on the category of spaces over BG, the functor $t \circ h^\infty \circ C_*$ is chain-equivalent to the functor C_* as sh-comodule functors over $\mathrm{S}' = \mathrm{H}_*(BG)$.* Thus, given the left G-space X, the S'-comodule $t^\infty(C_*(X))$, being a model for the chains of the Borel construction $N(G,X)$, cf. Theorem 4.6, calculates the G-equivariant homology of X; application of the functor h to the twisted object $t^\infty(C_*X)$ calculating the G-equivariant homology of X then reproduces an object calculating the ordinary homology of X. Likewise, given the space $Y \to BG$ over BG, the twisted object $h^\infty(C_*Y)$ is a model for the chains of the total space of the G-bundle over Y induced via $Y \to BG$ and application of the functor t to $h^\infty(C_*Y)$ reproduces an object calculating the ordinary homology of Y.

5^*. Koszul duality for cohomology

We maintain the circumstances of the previous section. Let $\mathrm{S} = \mathrm{Hom}(\mathrm{S}', \mathbb{R})$; since V is of finite type, S amounts to $\mathrm{S}[s^{-1}V^*]$, the symmetric algebra on the desuspension of the graded dual V^* of V. Likewise, $\Lambda' = \mathrm{Hom}(\Lambda, \mathbb{R})$ amounts to the exterior coalgebra cogenerated by V^*, and the induced twisting cochain $\tau^*\colon \Lambda' \to \mathrm{S}$ is acyclic. Non-positively graded objects will be indicated with the notation ${}_{\leq 0}$ if need be. For intelligibility, we note that *a (left) sh-module over* S *is a (left)* $(\overline{\Omega}\Lambda')$*-module*, and that *a (right) sh-module over* Λ *is a (right)* $(\overline{\Omega}\mathrm{S}')$*-module.* Consider the functors

$$(5.1^*) \qquad t^*_\infty\colon \mathrm{Mod}^\infty_\Lambda \to {}_{\mathrm{S}}\mathrm{Mod}, \quad t^*_\infty(N^\sharp, \tau_{\overline{\Omega}}) \;= \mathrm{Hom}^{\tau_{\overline{\Omega}}}(\mathrm{S}', N^\sharp),$$

$$(5.2^*) \qquad h^*_\infty\colon {}_{\mathrm{S}}\mathrm{Mod}^\infty \to \mathrm{Mod}_\Lambda, h^*_\infty(M^\sharp, \tau_{\overline{\Omega}\Lambda'}) = \Lambda' \otimes_{\tau_{\overline{\Omega}\Lambda'}} M^\sharp = \mathrm{Hom}^{\tau^{\overline{\mathrm{B}}\Lambda}}(\Lambda, M^\sharp),$$

where Λ' is a right Λ-module via the left Λ-module structure on itself, that is, Λ' is a right Λ-module via the operation of contraction. These functors restrict to functors

$$t^*_\infty\colon {}_{\leq 0}\mathrm{Mod}^\infty_\Lambda \to {}_{\mathrm{S}}\mathrm{Mod}_{\leq 0}, \quad h^*_\infty\colon {}_{\mathrm{S}}\mathrm{Mod}^\infty_{\leq 0} \to {}_{\leq 0}\mathrm{Mod}_\Lambda.$$

Proposition 5.3*. *The functors t^*_∞ and h^*_∞ are quasi-inverse to each other and yield a quasi-equivalence of categories between* $\mathrm{Mod}^\infty_\Lambda$ *and* ${}_{\mathrm{S}}\mathrm{Mod}^\infty$ *which restricts to a quasi-equivalence of categories between* ${}_{\le 0}\mathrm{Mod}^\infty_\Lambda$ *and* ${}_{\mathrm{S}}\mathrm{Mod}^\infty_{\le 0}$.

The meaning of this proposition, the significance of the terms "quasi-inverse" and "quasi-equivalence", and the proof of the proposition are somewhat similar to those for Proposition 5.3$_*$. We refrain from spelling out details.

5.4*. Koszul duality in ordinary equivariant cohomology, cf. [21], [22], [24]. As before, let G be a topological group of strictly exterior type such that the duals of the primitive generators are universally transgressive, let $\Lambda = \mathrm{H}_*G$, $\mathrm{S}' = \mathrm{H}_*(BG)$, $\Lambda' = \mathrm{H}^*G$, and $\mathrm{S} = \mathrm{H}^*(BG)$. *Any right (C^*G)-comodule is a right sh-comodule over* H^*G *via the twisting cochain* $\zeta^*_G\colon C^*G \to \mathrm{H}^*(BG)$, cf. (4.4.1*), and *any (left) $(\overline{\Omega}C^*G)$-module is an sh-module over* $\mathrm{H}^*(BG)$ via the twisting cochain $(\zeta^B)^*\colon \mathrm{H}^*G \to \overline{\Omega}C^*G$, cf. (4.4.1$_*$). In particular, C^*X is a right sh-module over H_*G via ζ^*_G and, for any space Y over BG, the cochains C^*Y constitute a (left) sh-module over $\mathrm{H}^*(BG)$ via $(\zeta^B)^*$ and the dual of the twisting cochain (4.2.1). Thus the functors (5.1*) and (5.2*) are defined, and the *statement of Proposition 5.3* entails that ordinary and equivariant cohomology are related by Koszul duality in the following sense: On the category of left G-spaces, the functor $h^* \circ t^*_\infty \circ C^*$ is chain-equivalent to the functor C^* as sh-comodule functors over $\Lambda' = \mathrm{H}^*G$; and on the category of spaces over BG, the functor $t^* \circ h^*_\infty \circ C^*$ is chain-equivalent to the functor C^* as sh-module functors over* $\mathrm{S} = \mathrm{H}^*(BG)$. Thus, given the G-space X, the twisted object $t^*_\infty(C^*X)$, being a model for the Borel construction, calculates the G-equivariant cohomology of X; application of the functor h^* to $t^*_\infty(C^*X)$ then reproduces an object calculating the ordinary cohomology of X. Likewise, the twisted object $h^*_\infty(C^*Y)$ is a model for the cochains of the total space of the induced G-bundle over Y and application of the functor t^* to $h^*_\infty(C^*Y)$ reproduces an object calculating the ordinary cohomology of Y.

6. Grand unification

Given two augmented differential graded algebras A_1 and A_2 and two sh-maps from A_1 to A_2, that is, twisting cochains $\tau_1, \tau_2\colon \overline{\mathrm{B}}A_1 \to A_2$, a *homotopy of sh-maps* from τ_1 to τ_2 is a homotopy of twisting cochains from τ_1 to τ_2. Likewise, given two coaugmented differential graded coalgebras C_1 and C_2 and two sh-maps from C_1 to C_2, that is, twisting cochains $\tau_1, \tau_2\colon C_1 \to \overline{\Omega}C_2$, a *homotopy of sh-maps* from τ_1 to τ_2 is a homotopy of twisting cochains from τ_1 to τ_2.

Recall that the category $\mathrm{DASH_h}$ has as its objects augmented differential non-negatively graded algebras with homotopy classes of sh-maps as morphisms; likewise the category $\mathrm{DCSH_h}$ has as its objects connected differential non-negatively graded coalgebras, necessarily coaugmented in a unique manner, with homotopy classes of sh-maps as morphisms. For these matters, cf. [51]. In that reference, the ground ring is a field, but that is not strictly necessary, under suitable additional hypotheses; for example, it suffices to require that the graded algebras and graded coalgebras under discussion be free as modules over the ground ring. Below, the requisite hypotheses will always tacitly be assumed to hold. The differential Tor can then be defined as a functor, to be written as shTor, on $\mathrm{DASH_h}$ and the differential Cotor can be defined as a functor on $\mathrm{DCSH_h}$, to be written as shCotor; likewise, the differential Ext can be defined as a functor, to be written as shExt, on $\mathrm{DASH_h}$. In other words, with the

appropriate interpretation of the term 'extension', this yields functorial extensions of the originalt Tor-, Cotor-, and Ext-functors, and the functoriality of those extensions is then referred to as *extended functoriality* [29].

We maintain the hypothesis that G is of strictly exterior type in such a way that the duals of the exterior homology generators are universally transgressive. In the category $\mathrm{DASH_h}$ of differential non-negatively graded algebras with homotopy classes of sh-maps as morphisms, the twisting cochain $\zeta_G\colon \mathrm{H}_*(BG) \to C_*G$ given as (4.4.1*) yields an isomorphism from H_*G onto C_*G. Likewise, in the category $\mathrm{DCSH_h}$ of connected differential non-negatively graded coalgebras with homotopy classes of sh-maps as morphisms, the twisting cochain $\zeta^B\colon \overline{\mathrm{B}}C_*G \to \mathrm{H}_*G$ given as (4.4.1$_*$) above yields an isomorphism from $\overline{\mathrm{B}}C_*G$ (notice that this coalgebra is connected!) onto $\mathrm{H}_*(BG)$. In view of the extended functoriality of the Tor- and Cotor-functors, the Koszul duality functor t^∞ defined on the category ${}_\Lambda\mathrm{Mod}^\infty$ of left sh-modules over Λ, cf. (5.1$_*$) above, is then equivalent to a corresponding functor t to be defined shortly on the category ${}_{C_*G}\mathrm{Mod}$ of ordinary differential graded left (C_*G)-modules, cf. (6.1$_*$) below, in the sense that the isomorphism in $\mathrm{DASH_h}$ between C_*G and Λ—natural in G—and the appropriate isomorphism between the target objects induce an equivalence between these two functors; likewise, the Koszul duality functor h^∞ defined on the category ${}_{\mathrm{S}'}\mathrm{Comod}^\infty$ of left sh-comodules over S', cf. (5.2$_*$) above, is then equivalent to a corresponding functor h to be defined shortly on the category ${}_{\overline{\mathrm{B}}C_*G}\mathrm{Comod}$ of ordinary differential graded left $(\overline{\mathrm{B}}C_*G)$-comodules, cf. (6.2$_*$) below, in the sense that the isomorphism in $\mathrm{DCSH_h}$ between $\overline{\mathrm{B}}C_*G$ and S'—natural in G—and the appropriate isomorphism between the target objects induce an equivalence between these two functors. Furthermore, in view of the extended functoriality of the Ext- and Tor-functors, the Koszul duality functor t^*_∞ defined on the category $\mathrm{Mod}^\infty_\Lambda$ of right sh-modules over Λ, cf. (5.1*), is then equivalent to a corresponding functor t^* to be defined shortly on the category Mod_{C_*G} of ordinary differential graded right (C_*G)-modules, cf. (6.1*) below, the notion of equivalence of functors being interpreted in formally the same way as explained above, with the requisite modifications being taken into account; likewise the Koszul duality functor h^*_∞ defined on the category ${}_\mathrm{S}\mathrm{Mod}^\infty$ of left sh-modules over S, cf. (5.2*), is then equivalent to a corresponding functor h^* to be defined shortly on the category ${}_{\overline{\mathrm{B}}^*C_*G}\widehat{\mathrm{Mod}}$ of complete differential graded left $(\overline{\mathrm{B}}^*C_*G)$-modules, cf. (6.2*) below, again the notion of equivalence of functors being interpreted in formally the same way as explained above, with the requisite modifications being taken into account.

We will now make this precise. The definitions of the shTor, shExt and shCotor will be given in (6.5)–(6.8) below.

In ordinary equivariant singular (co)homology, the situation of Section 3 arises in the following manner: Let $A = C_*G$, $C = \overline{\mathrm{B}}C_*G$, and $\tau = \tau_G\colon \overline{\mathrm{B}}C_*G \to C_*G$. The functors t, h, t^*, h^* now take the form

(6.1$_*$) $$t\colon {}_{C_*G}\mathrm{Mod} \to {}_{\overline{\mathrm{B}}C_*G}\mathrm{Comod}, \quad t(N) = \overline{\mathrm{B}}C_*G \otimes_\tau N$$

(6.2$_*$) $$h\colon {}_{\overline{\mathrm{B}}C_*G}\mathrm{Comod} \to {}_{C_*G}\mathrm{Mod}, \quad h(M) = C_*G \otimes_\tau M$$

(6.1*) $$t^*\colon \mathrm{Mod}_{C_*G} \to {}_{\overline{\mathrm{B}}^*C_*G}\widehat{\mathrm{Mod}}, \quad t^*(N) = \mathrm{Hom}^\tau(\overline{\mathrm{B}}C_*G, N)$$

(6.2*) $$h^*\colon {}_{\overline{\mathrm{B}}^*C_*G}\widehat{\mathrm{Mod}} \to \mathrm{Mod}_{C_*G}, \quad h^*(M^\flat) = \mathrm{Hom}^\tau(C_*G, M^\flat).$$

In particular, the twisted object $t(C_*X)$ calculates the G-equivariant homology of X, and $t^*(C^*X)$ calculates the G-equivariant cohomology of X. Likewise, given a space $Y \to BG$ over BG, its normalized chain complex C_*Y inherits a left $(\overline{\mathrm{B}}C_*G)$-comodule structure via the adjoint $\overline{\tau}$ of τ in a canonical manner, and the normalized cochain complex C^*Y inherits a complete $(\overline{\mathrm{B}}^*C_*G)$-module structure. The twisted object $h(C_*Y)$ calculates the differential graded

$$\mathrm{Cotor}^{\overline{\mathrm{B}}C_*G}(R, C_*Y) \tag{6.3$_*$}$$

and, in view of the discussion in (1.2) above, under appropriate finiteness assumptions, the twisted object $h^*(C^*Y)$ calculates the differential graded

$$\mathrm{Tor}_{\overline{\mathrm{B}}^*C_*G}(R, C^*Y). \tag{6.3*}$$

By the Eilenberg-Moore theorem [19], with reference to the fiber square

$$\begin{array}{ccc} E_Y & \longrightarrow & EG \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & BG, \end{array} \tag{6.4}$$

$\mathrm{Cotor}^{\overline{\mathrm{B}}C_*G}(R, C_*Y)$ calculates the homology of E_Y and $\mathrm{Tor}_{\overline{\mathrm{B}}^*C_*G}(R, C^*Y)$ calculates the cohomology of E_Y, compatibly with the bundle structures. This situation is *dual* to that of Theorem 4.1. The original approach of Eilenberg and Moore involves appropriate resolutions in the differential graded category.

Let N be a (C_*G)-module; then (N, ζ_G) is an sh-module over H_*G and, for the sake of consistency with the definitions in (5) above, we will write this sh-module as $(N, \tau_{\overline{\Omega}})$ so that N is considered as a differential graded $(\overline{\Omega}\mathrm{H}_*(BG))$-module via $\zeta_G \colon \mathrm{H}_*(BG) \to C_*G$, cf. (4.4.1*). The small model

$$t^\infty(N, \tau_{\overline{\Omega}}) = (\mathrm{H}_*(BG)) \otimes_{\zeta_G} N \tag{6.5}$$

defines the $\mathrm{shTor}^{\mathrm{H}_*G}(R, (N, \tau_{\overline{\Omega}}))$, and the morphism

$$\overline{\zeta}_G \otimes \mathrm{Id} \colon (\mathrm{H}_*(BG)) \otimes_{\zeta_G} N \longrightarrow (\overline{\mathrm{B}}C_*(G)) \otimes_{\tau_G} N$$

of twisted objects which, for $N = C_*X$, comes down to (4.6.1$_*$) above, makes explicit the isomorphism between $\mathrm{shTor}^{\mathrm{H}_*G}(R, (N, \tau_{\overline{\Omega}}))$ and $\mathrm{Tor}^{C_*G}(R, N)$ induced by the isomorphism in $\mathrm{DASH}_{\mathrm{h}}$ from H_*G onto C_*G.

Likewise, let M be a $(\overline{\mathrm{B}}C_*G)$-comodule; then (M, ζ^B) is an sh-comodule over $\mathrm{H}_*(BG)$. Write this sh-comodule as $(M, \tau^{\overline{\mathrm{B}}})$ so that M is considered as a (differential graded) $(\overline{\mathrm{B}}\mathrm{H}_*(BG))$-comodule via $\zeta^B \colon \overline{\mathrm{B}}C_*G \to \mathrm{H}_*G$, cf. (4.4.1$_*$). The twisted object

$$h^\infty(M, \tau^{\overline{\mathrm{B}}}) = (\mathrm{H}_*G) \otimes_{\zeta^B} M \tag{6.6}$$

defines the $\mathrm{shCotor}^{\mathrm{H}_*(BG)}(R, (M, \tau^{\overline{\mathrm{B}}}))$, and the two morphisms

$$\overline{\zeta}^B \otimes \mathrm{Id} \colon (\overline{\Omega}\,\overline{\mathrm{B}}C_*G) \otimes_{\tau_G} M \longrightarrow (\mathrm{H}_*G) \otimes_{\zeta^B} M$$
$$\overline{\tau}_{C_*G} \otimes \mathrm{Id} \colon (\overline{\Omega}\,\overline{\mathrm{B}}C_*G) \otimes_{\tau_G} M \longrightarrow (C_*G) \otimes_{\tau_G} M$$

of twisted objects make explicit the isomorphism

$$\mathrm{Cotor}^{\overline{\mathrm{B}}C_*G}(R,M) \to \mathrm{shCotor}^{\mathrm{H}_*(BG)}(R,(M,\tau^{\overline{\mathrm{B}}}))$$

induced by the isomorphism in $\mathrm{DCSH_h}$ from $\overline{\mathrm{B}}C_*G$ onto $\mathrm{H}_*(BG)$. Thus, in view of the extended functoriality of the Tor- and Cotor-functors, the two Koszul duality functors h^∞ and t^∞ are equivalent to the functors h and t, respectively. The duality between the functors t and h described in Theorem 3.5_* implies the duality for the functors h^∞ and t^∞ spelled out in Proposition 5.3_* and hence the Koszul duality in equivariant singular homology as given in (5.4_*) above.

In the same vein: Let N be a (C_*G)-module; then (N,ζ_G) is an sh-module over H_*G. Write this sh-module as $(N,\tau_{\overline{\Omega}})$ so that N is considered as a (differential graded) $(\overline{\Omega}\mathrm{H}_*(BG))$-module via ζ_G. The small model

$$t^*_\infty(N,\tau_{\overline{\Omega}}) = \mathrm{Hom}^{\tau_{\overline{\Omega}}}(\mathrm{S}',N) = \mathrm{Hom}^{\zeta_G}(\mathrm{H}_*(BG),N) \tag{6.7}$$

defines the $\mathrm{shExt}_{\mathrm{H}_*G}(R,(N,\tau_{\overline{\Omega}}))$, and the morphism

$$\mathrm{Hom}^{\zeta_G}(\mathrm{H}_*(BG),N) \leftarrow \mathrm{Hom}^{\tau_G}(\overline{\mathrm{B}}C_*(G),N)$$

of twisted Hom-objects which for $N = C^*X$ comes down to $(4.6.1^*)$ above, makes explicit the isomorphism between $\mathrm{shExt}_{\mathrm{H}_*G}(R,(N,\tau_{\overline{\Omega}}))$ and $\mathrm{Ext}_{C_*G}(R,N)$ induced by the isomorphism in $\mathrm{DASH_h}$ from H_*G onto C_*G. Likewise, let $M^\flat$ be a complete $(\overline{\mathrm{B}}^*C_*G)$-module; then (M,ζ^B) is an sh-module over $\mathrm{H}^*(BG)$. Write this sh-module as $(M^\flat,\tau_{\overline{\Omega}\Lambda'})$ so that $M^\flat$ is considered as a (differential graded) $(\overline{\Omega}\mathrm{H}^*G)$-module via ζ^B. More precisely, the adjoint

$$\overline{(\zeta^B)^*}\colon \overline{\Omega}\mathrm{H}^*G \to \overline{\mathrm{B}}^*C_*G$$

of the dual $(\zeta^B)^*\colon \mathrm{H}^*G \to \overline{\mathrm{B}}^*C_*G$ makes $M^\flat$ into a (differential graded) $(\overline{\Omega}\mathrm{H}^*G)$-module. The twisted object

$$h^*_\infty(M^\flat,\tau_{\overline{\Omega}\Lambda'}) = \Lambda' \otimes_{\tau_{\overline{\Omega}\Lambda'}} M^\flat = \mathrm{Hom}^{\tau^{\overline{\mathrm{B}}\Lambda}}(\Lambda,M^\flat) = \mathrm{Hom}^{\zeta^B}(\mathrm{H}_*G,M^\flat) \tag{6.8}$$

defines $\mathrm{shTor}_{\mathrm{H}^*(BG)}(R,(M^\flat,\tau_{\overline{\Omega}\Lambda'}))$ (since $\mathrm{H}^*(BG)$ is of finite type), and the two morphisms

$$\mathrm{Hom}^{\tau_G}(\overline{\Omega}\,\overline{\mathrm{B}}C_*G,M^\flat) \leftarrow \mathrm{Hom}^{\zeta^B}(\mathrm{H}_*G,M^\flat),$$
$$\mathrm{Hom}^{\tau_G}(\overline{\Omega}\,\overline{\mathrm{B}}C_*G,M^\flat) \leftarrow \mathrm{Hom}^{\tau_G}(C_*G,M^\flat)$$

of twisted Hom-objects, the former being induced by ζ^B and the latter being the obvious one, make explicit the isomorphism between $\mathrm{Tor}_{\overline{\mathrm{B}}^*C_*G}(R,M^\flat)$ and $\mathrm{shTor}_{\mathrm{H}^*(BG)}(R,(M^\flat,\tau_{\overline{\Omega}\Lambda'}))$ induced by the isomorphism in $\mathrm{DASH_h}$ from $\overline{\mathrm{B}}^*C_*G$ onto $\mathrm{H}^*(BG)$. Thus, in view of the extended functoriality of the Ext- and Tor-functors, the two Koszul duality functors h^*_∞ and t^*_∞ are equivalent to the functors h^* and t^*, respectively. The duality between the functors t^* and h^* described in Theorem 3.5^* implies the duality for the functors h^*_∞ and t^*_∞ spelled out in Proposition 5.3^*

and hence the Koszul duality in equivariant singular cohomology as given in (5.4*) above.

7. Some illustrations

Let G be a group, possibly endowed with additional structure (e. g. a topological group or a Lie group). Recall that a G-space is called *equivariantly formal* over the ground ring R when the spectral sequence from ordinary cohomology to G-equivariant cohomology collapses, cf. [2] and [20]. Thus when the G-space X is equivariantly formal (over R), the graded object $\mathrm{E}_\infty(X)$ associated with the G-equivariant cohomology $\mathrm{H}^*_G(X)$ relative to the Serre filtration (the filtration coming from $\mathrm{H}^*(BG)$-degree) takes the form of $\mathrm{H}^*(BG \times X)$ but this does not imply that the equivariant cohomology is, as a graded $\mathrm{H}^*(BG)$-module, an induced module of the kind $\mathrm{H}^*(BG, \mathrm{H}^*(X))$ (or of the kind $\mathrm{H}^*(BG) \otimes \mathrm{H}^*(X)$ when $\mathrm{H}^*(BG)$ is free over the ground ring) unless the additive extension problem is trivial, for example when the ground ring is a field. The property that the equivariant cohomology is such an induced module is strictly stronger than equivariant formality. See [23] Theorem 1.1 for a discussion and, in particular, Example 5.2 in that paper. Below we shall come back to the difference between the two properties.

According to an observation in [47] (proof of Proposition 6.8), a smooth compact symplectic manifold, endowed with a hamiltonian action of a compact Lie group, is equivariantly formal over the reals. This fact is also an immediate consequence of the result in [20] which says that, given a torus T and a compact hamiltonian T-manifold X, the real T-equivariant cohomology of X is an extended $\mathrm{H}^*(BT)$-module of the kind

$$\oplus \mathrm{H}^*(BT) \otimes \mathrm{H}^{*-\lambda_k}(F_k)$$

where $F_1, \ldots, F_s$ are the finitely many components of the fixed point set X^T and where $\lambda_1, \ldots, \lambda_s$ are even natural numbers. Here the compactness of the manifold is crucial as the following example shows, which also serves as an illustration for the notions of twisting cochain etc. exploited above:

Example 7.1

The familiar S^1-action on $X = \mathbb{C}^n \setminus 0$ is free and hence cannot be equivariantly formal. This action is plainly hamiltonian. The induced action of $\mathrm{H}_*(S^1)$ on $\mathrm{H}_*(X)$ is manifestly trivial and hence lifts to the trivial action of $\mathrm{H}_*(S^1)$ on $C_*(X)$. Over the integers $\mathbb{Z}$ as ground ring, we now describe the corresponding sh-action of $\mathrm{H}_*(S^1)$ on $C_*(X)$ which has to be non-trivial since the S^1-action on X cannot be equivariantly formal. This example is, admittedly, nearly a toy example, but it illustrates how under such circumstances homological perturbation theory works in general. In particular, it illustrates the fact spelled out already in the introduction that, for a general Lie group G and a general G-space Y, even when the induced action of $\mathrm{H}_*(G)$ on $\mathrm{H}_*(Y)$ lifts to an action of $\mathrm{H}_*(G)$ on $C_*(Y)$, in general only an sh-action of $\mathrm{H}_*(G)$ on $C_*(Y)$ involving higher degree terms will recover the geometry of the action.

As a coalgebra, the homology $\mathrm{H}_*(BS^1)$ is the symmetric coalgebra cogenerated by a single generator u of degree 2. This is the coalgebra which underlies the divided polynomial Hopf algebra $\Gamma = \Gamma[u]$ on u. This algebra, in turn, has a generator $\gamma_i(u)$ of degree $|\gamma_i(u)| = 2i$ in each degree $i \geq 1$ where $\gamma_1(u) = u$, subject to the relations

$$\gamma_i(u)\gamma_j(u) = \binom{i+j}{j}\gamma_{i+j}(u), \quad i, j \geq 1,$$

and the diagonal map Δ is given by the formula

$$\Delta\gamma_i(u) = \sum_{j+k=i} \gamma_j(u) \otimes \gamma_k(u), \quad i \geq 1.$$

Let

$$\tau\colon \Gamma[u] \longrightarrow \mathrm{End}(\mathrm{H}_*(S^{2n-1}))$$

be the twisting cochain which is zero on $\gamma_j(u)$ for $j \neq n$ and whose value $\tau(\gamma_n(u))$ is the endomorphism of $\mathrm{H}_*(S^{2n-1})$ which sends $1 \in \mathrm{H}_0(S^{2n-1})$ to one of the two generators of $\mathrm{H}_{2n-1}(S^{2n-1}) \cong \mathbb{Z}$. We assert that $\mathrm{H}_*(S^{2n-1})$ may be taken as a model for the chains on X and that the requisite sh-action of $\mathrm{H}_*(S^1)$ on C_*X then amounts to the twisting cochain τ.

In order to justify this claim, let v be a generator of $\mathrm{H}_1(S^1)$, write the homology algebra $\mathrm{H}_*(S^1)$ as the exterior algebra $\Lambda[v]$, and let $\vartheta\colon \Gamma[u] \to \Lambda[v]$ be the obvious acyclic twisting cochain which sends u to v and is zero on $\gamma_j(u)$ for $j \neq 1$. Let w be an indeterminate of degree 2 and denote by $\Gamma_{n-1}[w]$ the subcoalgebra of $\Gamma[w]$ which is additively generated by $\gamma_1(w), \ldots, \gamma_{n-1}(w)$. When $n = 1$, this coalgebra comes down to the ground ring which is here taken to be that of the integers. The coalgebra $\Gamma_{n-1}[w]$ amounts to the homology of complex projective $(n-1)$-space $\mathbb{C}P^{n-1}$, in fact, can be taken as a model for the chains of $\mathbb{C}P^{n-1}$: Take $C_*(\mathbb{C}P^{n-1})$ to be the normalized chain complex of the first Eilenberg subcomplex of $\mathbb{C}P^{n-1}$. A choice $\zeta\colon C_*(\mathbb{C}P^{n-1}) \to \mathbb{Z}$ of 2-cocycle representing the generator of $\mathrm{H}^2(\mathbb{C}P^{n-1})$ determines a twisting cochain $\tau\colon C_*(\mathbb{C}P^{n-1}) \longrightarrow \Lambda[v]$ such that, for degree reasons, the values of adjoint $\overline{\tau}\colon C_*(\mathbb{C}P^{n-1}) \longrightarrow \overline{\mathrm{B}}\Lambda[v] \cong \Gamma[u]$ of τ lie in $\Gamma_{n-1}[u]$, that is, the adjoint takes the form

$$\overline{\tau}\colon C_*(\mathbb{C}P^{n-1}) \longrightarrow \Gamma_{n-1}[u],$$

necessarily a chain equivalence, since $\overline{\tau}$ induces an isomorphism on homology and since the two chain complexes are free over the ground ring. Alternatively, the fact that the homology of complex projective $(n-1)$-space $\mathbb{C}P^{n-1}$, can be taken as a model for its chains results from the familiar cell decomposition of this space. Likewise the standard cell decomposition of the $(2n-1)$-sphere relative to the S^1-action has a single cell in each degree in such a way that the resulting cellular chain complex is exactly the chain complex which underlies the twisted tensor product

$$N = \Lambda[v] \otimes_\vartheta \Gamma_{n-1}[w]$$

where $\Gamma_{n-1}[w]$ is viewed as a right $\Gamma[w]$-comodule in the obvious manner; this chain complex is an equivariant model for the chains of the $(2n-1)$-sphere S^{2n-1} and hence of X as a free left S^1-space. We remind the reader that S^{2n-1} is the *Stiefel manifold* $G(n, n-1)$ of unitary complex 1-frames in $\mathbb{C}^n$; cell decompositions of general Stiefel manifolds can be found in [56] (Chap. 4).

Somewhat more formally: Pick a 1-cocycle representing the generator of $\mathrm{H}^1(S^1)$ and view this 1-cocycle as a morphism $\vartheta\colon C_*(S^1) \to \Lambda[v]$ of differential graded algebras, necessarily inducing an isomorphism on homology. The twisted Eilenberg-Zilber theorem, applied to the the principal S^1-bundle $S^{2n-1} \to \mathbb{C}P^{n-1}$, yields a twisting cochain $\tau^{\mathbb{C}P^{n-1}}\colon C_*(\mathbb{C}P^{n-1}) \to C_*(S^1)$ and a contraction

$$(C_*(S^1) \otimes_{\tau^{\mathbb{C}P^{n-1}}} C_*(\mathbb{C}P^{n-1}) \underset{\nabla}{\overset{\pi}{\leftrightarrows}} C_*(S^{2n-1}), h)$$

of $C_*(S^{2n-1})$ onto $C_*(S^1)\otimes_{\tau^{\mathbb C P^{n-1}}} C_*(\mathbb C P^{n-1})$ that is compatible with the $(C_*(S^1))$-module structures. Hence $C_*(S^1)\otimes_{\tau^{\mathbb C P^{n-1}}} C_*(\mathbb C P^{n-1})$ is a $(C_*(S^1))$-equivariant model for the chains of S^{2n-1}. The composite

$$\vartheta\tau^{\mathbb C P^{n-1}}\colon \mathbb C P^{n-1} \longrightarrow \Lambda[v]$$

is plainly a twisting cochain in such a way that

$$\Lambda[v]\otimes_{\vartheta\tau^{\mathbb C P^{n-1}}} C_*(\mathbb C P^{n-1})$$

is a $(C_*(S^1))$-equivariant model for the chains of S^{2n-1}. The twisting cochain $\vartheta\tau^{\mathbb C P^{n-1}}$ can be interpreted as a choice of 2-cocycle on $C_*(\mathbb C P^{n-1})$ of the kind ζ made above and, with this choice of ζ, the morphism

$$\mathrm{Id}\otimes\overline{\tau}\colon \Lambda[v]\otimes_{\vartheta\tau^{\mathbb C P^{n-1}}} C_*(\mathbb C P^{n-1}) \longrightarrow \Lambda[v]\otimes_{\vartheta}\Gamma_{n-1}[w]$$

of twisted tensor products, necessarily a chain equivalence and compatible with the requisite additional bundle structure, justifies the claim that $N=\Lambda[v]\otimes_{\vartheta}\Gamma_{n-1}[w]$ is an equivariant model for the chains of the $(2n-1)$-sphere S^{2n-1}.

As a chain complex, N decomposes canonically as $N = B\oplus \mathrm H_*(S^{2n-1})$ where B is contractible. Indeed, N has the additive generators $\gamma_k(w)$ and $v\otimes\gamma_k(w)$ where $0\le k\le n-1$ and

$$d\gamma_k(w) = v\otimes\gamma_{k-1}(w),\quad dv\otimes\gamma_k(w)=0,\quad 0\le k\le n-1.$$

Now B is the span of

$$v, w, v\otimes w, \gamma_2(w),\dots, v\otimes\gamma_{n-2}(w), \gamma_{n-1}(w)$$

and $\mathrm H_*(S^{2n-1})$ is additively generated by $1=\gamma_0(w)$ and $v\otimes\gamma_{n-1}(w)$.

Consider, then, the twisted tensor product

$$\Gamma[u]\otimes_{\vartheta} N = \Gamma[u]\otimes_{\vartheta}\Lambda[v]\otimes_{\vartheta}\Gamma_{n-1}[w].$$

It calculates the S^1-equivariant cohomology of S^{2n-1} and, indeed, the homology of this twisted tensor product plainly amounts to $\Gamma_{n-1}[w]$, the homology of complex projective $(n-1)$-space.

The obvious surjection from N onto $\mathrm H_*(S^{2n-1})$ extends to a contraction

$$(\mathrm H_*(S^{2n-1}) \underset{\nabla}{\overset{\pi}{\rightleftarrows}} N, h)$$

in an obvious manner. Tensoring this contraction with $\Gamma[u]$, we obtain the contraction

$$(\Gamma[u]\otimes \mathrm H_*(S^{2n-1}) \underset{\mathrm{Id}\otimes\nabla}{\overset{\mathrm{Id}\otimes\pi}{\rightleftarrows}} \Gamma[u]\otimes N, \mathrm{Id}\otimes h).$$

Write the differential on $\Gamma[u]\otimes_{\vartheta} N$ as $d+\partial$ where the perturbation ∂ arises from the twisting cochain ϑ, that is, this perturbation equals the operator $\vartheta\cap\cdot$. Application of the perturbation lemma (Lemma 2.3) yields a perturbation $\mathcal D$ of the zero differential together with a contraction

$$(\Gamma[u]\otimes \mathrm H_*(S^{2n-1}),\mathcal D) \underset{\nabla_{\partial}}{\overset{g_{\partial}}{\rightleftarrows}} \Gamma[u]\otimes_{\vartheta} N, h_{\partial}).$$

Evaluating the formula (2.3.1) under the present circumstances we see that the perturbation $\mathcal D$ is the twisted differential associated with the twisting cochain τ, that is,

$$(\Gamma[u]\otimes \mathrm H_*(S^{2n-1}),\mathcal D) = \Gamma[u]\otimes_{\tau}\mathrm H_*(S^{2n-1}).$$

Hence $\mathrm H_*(S^{2n-1})$ may be taken as a model for the chains on X, and the sh-action of $\mathrm H_*(S^1)$ on C_*X then amounts to the twisting cochain τ from $\Gamma[u]$ to $\mathrm{End}(\mathrm H_*(S^{2n-1}))$.

Example 7.2

Consider a homogeneous space G/K of a compact Lie group G by a closed subgroup K. As noted earlier, the cohomology of G/K equals the K-equivariant cohomology of G. Let R be a ground ring such that G and K are both of strictly exterior type over R. The familiar model

$$\mathrm{H}_*(BK) \otimes_\tau \mathrm{H}_*G$$

for the homology of G/K, cf. e. g. [46], incorporates the requisite sh-action of H_*K on H_*G via the composite of the universal transgression for the universal G-bundle with the induced morphism from $\mathrm{H}_*(BK)$ to $\mathrm{H}_*(BG)$. In particular, the induced K-action on the total space T^*G of the cotangent bundle on G is hamiltonian, and the reduced space is the total space $\mathrm{T}^*(G/K)$ of the cotangent bundle on G/K. Thus, the (hamiltonian) K-action on T^*G cannot be equivariantly formal. This shows once again that, in order for a hamiltonian action of a compact Lie group to be equivariantly formal, the compactness of the manifold is crucial.

An explicit (admittedly toy) example arising from $G = \mathrm{SU}(2)$ and $K = T$, a maximal torus which here comes down to the circle group S^1, has been spelled out in the introduction and in Example 7.1 above.

Example 7.3

Let K be a connected and simply connected compact Lie group, and let $G = \mathrm{Map}^0(S^2, K)$, otherwise known as $\Omega^2 K$, the group of based smooth maps from S^2 to K, with pointwise multiplication. Endowed with the induced differentiable structure [10], [11], G is a Lie group. Let $x_1, \ldots, x_n$ be odd degree real cohomology classes of K so that the real cohomology H^*K is the exterior algebra on these classes. Then there are real cohomology classes $\xi_1, \ldots, \xi_n$ of G with $|\xi_j| = |x_j| - 2$ for $1 \leq j \leq n$ so that, when $x_1, \ldots, x_n$ are interpreted as cohomology classes of G in the obvious manner, the real cohomology H^*G is the exterior algebra on $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$; see e. g. [1] or [35] where such generators are constructed for a group of the kind $\mathrm{Map}^0(\Sigma, K)$ where Σ is a closed surface of arbitrary genus rather than just the 2-sphere S^2 (for genus higher than zero additional generators are necessary). Our approach to Koszul duality applies to the example $G = \mathrm{Map}^0(S^2, K)$.

Example 7.4

Let G be a Lie group of strictly exterior type over the ground ring R in such a way that the duals of the exterior generators are universally transgressive. Then, with respect to the conjugation action of G on itself, G is *equivariantly formal*, in fact, its equivariant cohomology is an induced module of the kind $\mathrm{H}^*(G) \otimes \mathrm{H}^*(BG)$. Indeed, consider the path fibration $\Omega G \to PG \to G$; with pointwise multiplication on ΩG and PG, the path fibration is an extension of groups. Moreover, with reference to the G-action on ΩG induced by conjugation, as a group, the free loop group decomposes as the semi-direct product $\Lambda G \cong \Omega G \rtimes G$. Now $\Omega G \rtimes G$ acts on PG where ΩG acts via translation and G by conjugation, and this action extends to a free action on $PG \times EG$ through the projection to G whence the Borel construction $G \times_G EG$ is a classifying space for ΛG. This classifying space, in turn, amounts to

the free loop space ΛBG on the classifying space BG. Over the cohomology $\mathrm{H}^*(BG)$ of the classifying space BG, the cohomology of ΛBG is an induced module of the kind $\mathrm{H}^*(G)\otimes \mathrm{H}^*(BG)$.

8. Split complexes and generalized momentum mapping

Let C be a coaugmented differential graded coalgebra and A an augmented differential graded algebra.

We will refer to a twisting cochain which is homotopic to the zero twisting cochain as an *exact* twisting cochain; thus τ being exact means that there is a morphism $h\colon C \to A$ of degree zero such that

$$Dh = \tau \cup h, \quad h\eta = \eta, \quad \varepsilon h = \varepsilon. \tag{8.1}$$

Such a morphism h will then be referred to as a *splitting* for τ.

Let $\tau\colon C\to A$ be a twisting cochain, and let N be a differential graded right A-module. Consider the twisted Hom-object $\mathrm{Hom}^\tau(C,N)$. We will say that this object is *split* when the composite of the twisting cochain τ with the action $A\to \mathrm{End}(N)^{\mathrm{op}}$ is exact, that is, when there is a morphism $h\colon C\to \mathrm{End}(N)^{\mathrm{op}}$ of degree zero satisfying (8.1), with $\mathrm{End}(N)^{\mathrm{op}}$ substituted for A. We then refer to h as a *splitting homotopy* for $\mathrm{Hom}^\tau(C,N)$. The notion of split object arises by abstraction from the property of a G-space having the property that its equivariant cohomology is an induced module over $\mathrm{H}^*(BG)$, that is, makes precise the idea that the action behaves like the trivial action, as far as equivariant cohomology is concerned.

Proposition 8.2. *Suppose that C is cocomplete. Given a splitting homotopy h for the twisted Hom-object $\mathrm{Hom}^\tau(C,N)$, the assignment to a (homogeneous) $\phi\in\mathrm{Hom}(C,N)$ of $\phi\cup h\in \mathrm{Hom}(C,N)$ yields an isomorphism*

$$\mathrm{Hom}^\tau(C,N)\longrightarrow \mathrm{Hom}(C,N)$$

of twisted Hom-*objects, the target of the isomorphism being untwisted.*

Proof. This is a special case of Lemma 1.2.1 above. □

Thus a twisted Hom-object which is split is isomorphic to the corresponding untwisted object.

Suppose that, as a graded R-modules, C is free and that N is the dual of a free graded R-module. Consider a general twisted Hom-object $\mathrm{Hom}^\tau(C,N)$. In the standard manner, the filtration by C-degree yields a spectral sequence (E_r,d_r) $(r\geq 0)$ which has

$$(\mathrm{E}_0,d_0)\cong(\mathrm{Hom}(C,N),d_N),\ \ (\mathrm{E}_1,d_1)\cong(\mathrm{Hom}(C,\mathrm{H}^*N),d_C),\ \ \mathrm{E}_2\cong \mathrm{H}^*(C,\mathrm{H}^*N).$$

Corollary 8.3. *When the twisted Hom-object $\mathrm{Hom}^\tau(C,N)$ is split the spectral sequence (E_r,d_r) $(r\geq 0)$ collapses from E_2.*

Proof. This is an immediate consequence of Proposition 8.2. □

Let $N^\sharp$ be a differential graded left A-module whose dual $(N^\sharp)^*$ coincides with the differential graded right A-module N and suppose that $N^\sharp$ is non-negative and connected. Suppose that there is a contraction $(\mathrm{H}_*(N^\sharp) \underset{\nabla^\sharp}{\overset{\pi^\sharp}{\leftrightarrows}} N^\sharp, h^\sharp)$ of chain complexes. Notice that, when the ground ring R is a field, such a contraction necessarily exists.

Theorem 8.4. *Under these circumstances, the twisted Hom-object* $\mathrm{Hom}^\tau(C,N)$ *is split if and only if the spectral sequence* (E_r,d_r) $(r\geq 0)$ *collapses from* E_2.

Thus, over a field, the collapse of the spectral sequence from E_2 suffices to guarantee that the twisted Hom-object is split.

We begin with the preparations for the proof of Theorem 8.4. Consider the differential graded algebra $\mathrm{End}(N^\sharp)$ of homogeneous endomorphisms of $N^\sharp$, endowed with the obvious differential graded algebra structure, and let $\mathcal{A}=\mathrm{End}(N^\sharp)_{\geq 0}$ be the differential graded subalgebra of homogeneous endomorphisms of non-negative degree. The assignment to a degree zero endomorphism of $N^\sharp$ of its degree zero constituent $N^\sharp_0\to N^\sharp_0$ yields a multiplicative augmentation map $\varepsilon\colon\mathcal{A}\to R$ for $\mathcal{A}$. Likewise consider the graded algebra $\mathrm{End}(\mathrm{H}_*(N^\sharp))$ of homogeneous endomorphisms of $\mathrm{H}_*(N^\sharp)$, endowed with the obvious graded algebra structure, and let $\mathcal{B}=\mathrm{End}(\mathrm{H}_*(N^\sharp))_{\geq 0}$ be the graded subalgebra of homogeneous endomorphisms of non-negative degree. Similarly as before, the assignment to a degree zero endomorphism of $\mathrm{H}_*(N^\sharp)$ of its degree zero constituent $\mathrm{H}_0(N^\sharp)\to\mathrm{H}_0(N^\sharp)$ yields a multiplicative augmentation map $\varepsilon\colon\mathcal{B}\to R$ for $\mathcal{B}$. Moreover, the assigment to $\alpha\in\mathcal{A}$ of $\Pi(\alpha)=\pi^\sharp\alpha\nabla^\sharp\in\mathcal{B}$ and to $\beta\in\mathcal{B}$ of $\nabla(\beta)=\nabla^\sharp\beta\pi^\sharp\in\mathcal{A}$ yields chain maps

$$\Pi\colon\mathcal{A}\to\mathcal{B},\ \ \nabla\colon\mathcal{B}\to\mathcal{A}$$

such that $\Pi\nabla=\mathrm{Id}_{\mathcal{B}}$ but neither Π nor ∇ are morphisms of differential graded algebras unless the differential of $N^\sharp$ is zero.

Thus suppose that the differential of $N^\sharp$ is non-zero. While the projection Π is not a morphism of differential graded algebras, using HPT, we will now extend Π to an sh-morphism from $\mathcal{A}$ to $\mathcal{B}$, that is, to a twisting cochain $\tau_\Pi\colon\overline{\mathrm{B}}\mathcal{A}\to\mathcal{B}$.

For greater clarity, we proceed somewhat more generally. Thus let U be a general augmented differential graded algebra. Recall that, by construction, as a graded coalgebra, $\overline{\mathrm{B}}U$ is the graded tensor coalgebra

$$\overline{\mathrm{B}}U=\mathrm{T^c}[sIU]=\oplus(sIU)^{\otimes j}.$$

The differential $d_{\overline{\mathrm{B}}U}$ on $\overline{\mathrm{B}}U$ takes the form

$$d_{\overline{\mathrm{B}}U}=d_U+\partial.$$

Here d_U refers to the differential induced by the differential on U, that is, for $j\geq 1$, the operator $d_{\overline{\mathrm{B}}U}$ is defined on each homogeneous constituent $(sIU)^{\otimes j}$ and then takes the form

$$d_U\colon(sIU)^{\otimes j}\longrightarrow(sIU)^{\otimes j}$$

of the tensor product differential on the j-fold tensor product $(sIU)^{\otimes j}$ of the suspension sIU; thus, with the aid of the standard notation

$$[\alpha_1|\alpha_2|\dots|\alpha_j]=(s\alpha_1)\otimes\dots\otimes(s\alpha_j)\ \ (\alpha_\nu\in U)$$

and $\overline{\alpha}=(-1)^{|\alpha|+1}=\alpha$ $(\alpha\in U)$, the values of d_U are given by

$$\begin{aligned}d_U[\alpha]&=-[d(\alpha)]\\ d_U\,[\alpha_1|\dots|\alpha_j]&=-\sum[\overline{\alpha}_1|\dots|\overline{\alpha}_{\nu-1}|d\alpha_\nu|\alpha_{\nu+1}|\dots|\alpha_j]\,.\end{aligned}$$

Moreover, ∂ is the coalgebra perturbation (relative to the bar construction or coaugmentation filtration) determined by the multiplication in U, that is, ∂ is the coderivation of $\mathrm{T}^{\mathrm{c}}[sIU]$ determined by the requirement that the diagram

$$\begin{array}{ccc}
(sIU)\otimes(sIU) & \xrightarrow{\partial} & (sIU) \\
{\scriptstyle s^{-1}\otimes s^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle s} \\
(IU)\otimes(IU) & \xrightarrow{\mu} & (IU)
\end{array}$$

be commutative. Explicitly, ∂ is zero on sIU and, for $j\geq 2$, the values of ∂ are given by the identity

$$\partial\,[\alpha_1|\dots|\alpha_j] = \sum\,[\overline{\alpha}_1|\dots|\overline{\alpha}_{\nu-1}|\overline{\alpha}_\nu\alpha_{\nu+1}|\alpha_{\nu+2}|\dots|\alpha_j]\,.$$

Let $\tau_1 = \tau^{\overline{\mathrm{B}}U}\colon \overline{\mathrm{B}}U \to U$, the universal bar construction twisting cochain for U. Recall that τ_1 is the desuspension on sIU and zero elsewhere. Henceforth we will occasionally write the multiplication in U as $\cdot\colon U\otimes U\to U$, and we denote by D the unperturbed Hom-differential on $\mathrm{Hom}(\overline{\mathrm{B}}U,U)$, that is, the differential that relies only on the differential on U; then the perturbed differential on $\mathrm{Hom}(\overline{\mathrm{B}}U,U)$ which takes into account the multiplication $\cdot$ on U takes the form $D+\delta$ where $\delta(\varphi) = (-1)^{|\varphi|+1}\varphi\partial$ ($\varphi\in\mathrm{Hom}(\overline{\mathrm{B}}U,U)$).

By construction, τ_1 is a cycle relative to the differential d_U, that is,

$$D\tau_1 = d\tau_1 + \tau_1 d_U = 0. \tag{8.5}$$

Furthermore, the twisting cochain property of τ_1 is equivalent to the identity

$$\tau_1\partial = \tau_1\cup\tau_1\colon \overline{\mathrm{B}}U \longrightarrow U. \tag{8.6}$$

Let $p\in U$ be an idempotent, necessarily of degree zero. Using the idempotent p of U, we introduce a new composition $\circ\colon U\otimes U\to U$ in U by the assignment to (α,β) of

$$\alpha\circ\beta = \alpha p\beta,\quad \alpha,\beta\in U.$$

A little thought reveals that this composition is associative, that is, given $\alpha,\beta,\gamma\in U$,

$$(\alpha\circ\beta)\circ\gamma = \alpha\circ(\beta\circ\gamma).$$

Let $\odot$ denote the cup pairing in $\mathrm{Hom}(\overline{\mathrm{B}}U,U)$ relative to the original coalgebra structure on $\overline{\mathrm{B}}U$ but relative to the new composition $\circ$ on U. Thus, given $\alpha,\beta\in U$, the element $\alpha\odot\beta$ of $\mathrm{Hom}(\overline{\mathrm{B}}U,U)$ is given as the composite

$$\alpha\odot\beta\colon \overline{\mathrm{B}}U \xrightarrow{\Delta} \overline{\mathrm{B}}U\otimes\overline{\mathrm{B}}U \xrightarrow{\alpha\otimes\beta} U\otimes U \xrightarrow{\circ} U.$$

Lemma 8.7. *Given the idempotent p of U, let $h \in U$ be an element of degree 1 such that $dh = 1 - p$. Let $\tau_1 = \tau^{\overline{\mathrm{B}}U} \colon \overline{\mathrm{B}}U \to U$ and, for $j \geq 2$, define $\tau_j \colon (sIU)^{\otimes j} \to U$ by*

$$\tau_j\,[\alpha_1|\ldots|\alpha_j] = \alpha_1 \cdot h \cdot \alpha_2 \cdot \ldots \cdot \alpha_{j-1} \cdot h \cdot \alpha_j \quad (j-1 \text{ copies of } h).$$

Suppose that U is complete. Then

$$\tau^h = \tau_1 + \tau_2 + \ldots \colon \overline{\mathrm{B}}U \to (U, \circ)$$

is a twisting cochain with respect to the original coalgebra structure on $\overline{\mathrm{B}}U$ and the new composition $\circ$ on U, that is, τ satisfies the identity

$$(D+\delta)(\tau)(= D(\tau) + \tau\partial) = \tau \odot \tau.$$

Proof. Let $j \geq 2$. By construction

$$\begin{aligned}
(D\tau_j)\,[\alpha_1|\ldots|\alpha_j] &= d(\alpha_1 \cdot h \cdot \ldots \cdot \alpha_j) + \tau_j d_U\,[\alpha_1|\ldots|\alpha_j] \\
&= (d\alpha_1) \cdot h \cdot \ldots \cdot \alpha_j + \ldots \\
&\quad - \overline{\alpha}_1(1-p) \cdot \alpha_2 \cdot \ldots \cdot h \cdot \alpha_j \\
&\quad - \overline{\alpha}_1 \cdot h \cdot \overline{\alpha}_2 \cdot (1-p) \ldots \cdot h \cdot \alpha_j + \ldots \\
&\quad - \overline{\alpha}_1 \cdot h \cdot \overline{\alpha}_2 \cdot h \ldots \cdot h \cdot \overline{\alpha}_{j-1} \cdot (1-p)\alpha_j \\
&\quad - \tau_j \sum [\overline{\alpha}_1|\ldots|\overline{\alpha}_{\nu-1}|d\alpha_\nu|\alpha_{\nu+1}|\ldots|\alpha_n] \\
&= \sum_{\nu=1}^{j-1} \overline{\alpha}_1 \cdot h \cdot \ldots \cdot h \cdot \overline{\alpha}_\nu \cdot (p-1) \cdot \alpha_{\nu+1} \cdot h \cdot \ldots \cdot h \cdot \alpha_{j-1} \cdot h \cdot \alpha_j \\
&= \sum_{\nu=1}^{j-1} (\overline{\alpha}_1 \cdot h \cdot \ldots \cdot h \cdot \overline{\alpha}_\nu) \circ (\alpha_{\nu+1} \cdot h \cdot \ldots \cdot h \cdot \alpha_{j-1} \cdot h \cdot \alpha_j) \\
&\quad - \sum_{\nu=1}^{j-1} (\overline{\alpha}_1 \cdot h \cdot \ldots \cdot h \cdot \overline{\alpha}_\nu) \cdot (\alpha_{\nu+1} \cdot h \cdot \ldots \cdot h \cdot \alpha_{j-1} \cdot h \cdot \alpha_j)\,.
\end{aligned}$$

Likewise, let $1 \leq \nu \leq j-1$. Since

$$\begin{aligned}
(\tau_\nu \cup \tau_{j-\nu})\,[\alpha_1|\ldots|\alpha_j] &= (\tau_\nu\,[\overline{\alpha}_1|\ldots|\overline{\alpha}_\nu]) \cdot (\tau_{j-\nu}\,[\alpha_{n+1}|\ldots|\alpha_j]) \\
(\tau_\nu \odot \tau_{j-\nu})\,[\alpha_1|\ldots|\alpha_j] &= (\tau_\nu\,[\overline{\alpha}_1|\ldots|\overline{\alpha}_\nu]) \circ (\tau_{j-\nu}\,[\alpha_{n+1}|\ldots|\alpha_j])
\end{aligned}$$

we conclude

$$D\tau_j = \sum_{\nu=1}^{j-1} \tau_\nu \odot \tau_{j-\nu} - \sum_{\nu=1}^{j-1} \tau_\nu \cup \tau_{j-\nu}. \tag{8.7.1}$$

The same kind of calculation yields

$$\tau_{j-1}\partial = \sum_{\nu=1}^{j-1} \tau_\nu \cup \tau_{j-\nu} \tag{8.7.2}$$

Indeed,

$$
\begin{aligned}
\tau_{j-1}\partial\,[\alpha_1|\dots|\alpha_j] &= \sum \tau_{j-1}\,[\overline{\alpha}_1|\dots|\overline{\alpha}_{\nu-1}|\overline{\alpha}_\nu\alpha_{\nu+1}|\alpha_{\nu+2}|\dots|\alpha_j]\\
&= \sum \left(\overline{\alpha}_1\cdot h\cdot\ldots\cdot h\cdot\overline{\alpha}_{\nu-1}\cdot h\cdot\overline{\alpha}_\nu\right)\left(\alpha_{\nu+1}\cdot h\cdot\alpha_{\nu+2}\cdot h\cdot\ldots\cdot h\cdot\alpha_j\right)\\
&= \sum_{\nu=1}^{j-1}\tau_\nu\cup\tau_{j-\nu}\,[\alpha_1|\dots|\alpha_j]\,.
\end{aligned}
$$

Combining the identities (8.7.1) and (8.7.2) with the twisting cochain property of τ_1, cf. (8.5) and (8.6), we conclude

$$D(\tau_1+\tau_2+\dots)+(\tau_1+\tau_2+\dots)\partial=(\tau_1+\tau_2+\dots)\odot(\tau_1+\tau_2+\dots)$$

as asserted. □

We now apply Lemma 8.7 to the differential graded algebra $\mathcal{A}$ together with the idempotent $p=\nabla^\sharp\pi^\sharp\in\mathcal{A}$ and the homotopy $h^\sharp\in\mathcal{A}$. By construction,

$$\tau_\Pi=\Pi\tau^{h^\sharp}\colon\overline{\mathrm{B}}\mathcal{A}\to\mathcal{B}$$

is a twisting cochain which extends the projection $\Pi\colon\mathcal{A}\to\mathcal{B}$.

The original A-action on $N^\sharp$ amounts to a morphism $A\to\mathcal{A}$ of differential graded algebras, and the composite

$$C\xrightarrow{\tau}A\to\mathcal{A}\tag{8.8}$$

is a twisting cochain. Likewise the composite

$$C\to\overline{\mathrm{B}}\mathcal{A}\xrightarrow{\tau_{\overline{\Omega}}}\overline{\Omega\mathrm{B}}\mathcal{A}$$

of the adjoint of (8.8) with the universal cobar construction twisting cochain $\tau_{\overline{\Omega}}$ is a twisting cochain, and we can substitute $\overline{\Omega\mathrm{B}}\mathcal{A}$ for A and simply write $\tau\colon C\to\overline{\Omega\mathrm{B}}\mathcal{A}$, that is, we consider $N^\sharp$ as a differential graded left $\overline{\Omega\mathrm{B}}\mathcal{A}$-module and N as a differential graded right $\overline{\Omega\mathrm{B}}\mathcal{A}$-module through the adjunction map $\overline{\Omega\mathrm{B}}\mathcal{A}\longrightarrow\mathcal{A}$.

The adjoint

$$\overline{\tau_\Pi}\colon\overline{\Omega\mathrm{B}}\mathcal{A}\longrightarrow\mathcal{B}$$

of the twisting cochain τ_Π is a surjective morphism of differential graded algebras which extends to a contraction

$$(\overline{\Omega\mathrm{B}}\mathcal{A}\underset{\overline{\tau_\Pi}}{\overset{\nabla^\flat}{\leftrightarrows}}\mathcal{B},h^\flat).$$

Lemma 8.9. *Suppose that the following data are given:*
— augmented differential graded algebras A and B;
— a contraction $(B\underset{\nabla}{\overset{\pi}{\leftrightarrows}}A,h)$ of chain complexes, π being a morphism of augmented differential graded algebras;
— a coaugmented differential graded coalgebra C;

— *twisting cochains* $t_1, t_2 \colon C \to A$;
— *a homotopy* $h^B \colon C \to B$ *of twisting cochains* $h^B \colon \pi t_1 \simeq \pi t_2$, *so that*

$$D(h^B) = (\pi t_1) \cup h^B - h^B \cup (\pi t_2). \tag{8.9.1}$$

Suppose that C *is cocomplete. Then the recursive rule*

$$h^A = \nabla h^B - h(t_1 \cup h^A - h^A \cup t_2) \tag{8.9.2}$$

yields a homotopy $h^A \colon C \to A$ *of twisting cochains* $h^A \colon t_1 \simeq t_2$ *such that* $\pi h^A = h^B$.

The rule (8.9.2) being recursive means that

$$h^A = \varepsilon\eta + h_1 + h_2 + \dots$$

where $h_1 = \nabla h^B - h(t_1 - t_2)$, $h_2 = -h(t_1 \cup h_1 - h_1 \cup t_2)$, etc.

Proof. The identity $\pi h^A = h^B$ is obvious and, since t_1 and t_2 are ordinary twisting cochains, the morphism $t_1 \cup h^A - h^A \cup t_2$ is (easily seen to be) a cycle. Furthermore, since π is compatible with the algebra structures,

$$\begin{aligned}\nabla\pi(t_1 \cup h^A - h^A \cup t_2) &= \nabla((\pi t_1) \cup (\pi h^A) - (\pi h^A) \cup (\pi t_2))\\ &= \nabla((\pi t_1) \cup h^B - h^B \cup (\pi t_2)).\end{aligned}$$

Consequently

$$\begin{aligned}Dh^A &= \nabla(D(h^B)) + Dh(t_1 \cup h^A - h^A \cup t_2)\\ &= \nabla((\pi t_1) \cup h^B - h^B \cup (\pi t_2)) + (t_1 \cup h^A - h^A \cup t_2) - \nabla\pi(t_1 \cup h^A - h^A \cup t_2)\\ &= t_1 \cup h^A - h^A \cup t_2\end{aligned}$$

as asserted. $\square$

Proof of Theorem 8.4. Corollary 8.3 says that the condition is necessary. To justify that the condition is also sufficient we note first that the hypothesis that the spectral sequence collapses implies that the twisting cochain that arises as the composite

$$C \xrightarrow{\tau} \overline{\Omega\mathrm{B}}\mathcal{A} \xrightarrow{\overline{\tau_{\Pi}}} \mathcal{B}$$

of the twisting cochain $\tau \colon C \to \overline{\Omega\mathrm{B}}\mathcal{A}$ with the adjoint $\overline{\tau_{\Pi}}$ is the zero twisting cochain. Lemma 8.9, applied with $B = \mathcal{B}$, $A = \overline{\Omega\mathrm{B}}\mathcal{A}$, $t_1 = \tau$, t_2 the zero twisting cochain, and h^B the zero homotopy, then yields a splitting $h \colon C \to \overline{\Omega\mathrm{B}}\mathcal{A}$ for τ so that $Dh = \tau \cup h$, $h\eta = \eta$ and $\varepsilon h = \varepsilon$, cf. (8.1). $\square$

Remark 8.10. Theorem 8.4 does not contradict Example 5.2 in [23]. To adjust the present notation to that example, let $C = \mathrm{H}^*(BS^1)$, let $\mathcal{C}Y$ denote the cone on the space Y, let $X = S^2 \cup_\phi (\mathcal{C}\mathbb{RP}^2 \times S^1)$ be the space explored in [23], ϕ being the map $\mathbb{RP}^2 \times S^1 \xrightarrow{\mathrm{pr}} \mathbb{RP}^2 \to \mathbb{RP}^2/\mathbb{RP}^1 \cong S^2$, and let $N = C^*(X)$. (We use the font $\mathcal{C}$ to distinguish the cone from our notation C for a differential graded coalgebra.) The space X has $\mathrm{H}^0(X)$ and $\mathrm{H}^2(X)$ infinite cyclic, $\mathrm{H}^4(X)$ cyclic of order 2, and its

other cohomology groups are zero. Plainly there is no way to contract $C_*(X)$ onto $\mathrm{H}_*(X)$, so there is no contradiction. In fact, under such circumstances, the existence of the contraction implies that the equivariant cohomology is an extended module once the spectral sequence collapses. On the other hand, the Example 5.2 in [23] shows that the collapse of the spectral sequence from E_2 alone does not guarantee that the equivariant cohomology is an extended module. I am indebted to the referee for having insisted that this point be clarified.

8.11. Splitting homotopy and momentum mapping. We will now explain in which sense a splitting homotopy generalizes a momentum mapping: As before, let G be a topological group of strictly exterior type, let $C = \overline{\mathrm{B}}C_*G$, $A = C_*G$, $\tau = \tau_G \colon \overline{\mathrm{B}}C_*G \to C_*G$, let X be a left G-space, and let $N = C^*(X)$, viewed as a right (C_*G)-module as above. In view of Lemma 8.3, Proposition 9.3 in [24] shows that our notion of split twisted Hom-object is consistent with the notion of split complex used in [24]. Consider a 2-cocycle ζ of X. In the double complex (E_2, d_2), the class $[\zeta] \in \mathrm{H}^2(X)$ lies on the fiber line, in fact, in $\mathrm{E}_2^{0,2}$. Since G is of striclty exterior type, $d_2[\zeta] \in \mathrm{E}_2^{2,1}$ is the obstruction to the existence of an equivariantly closed extension of ζ, that is, the obstruction to the existence of a pre-image in $\mathrm{H}^2_G(X)$ relative to the restriction mapping $\mathrm{H}^2_G(X) \to \mathrm{H}^2(X)$. This obstruction comes down to the obstruction to the existence of a $C^0(X)$-valued 2-cochain φ on $\overline{\mathrm{B}}C_*G$ (where $C^0(X)$ refers to the singular zero cochains on X) such that

$$d \circ \varphi = \zeta \cup \tau \colon \overline{\mathrm{B}}C_*G \longrightarrow C^1(X) \tag{8.11.1}$$

where d is the differential on $C^*(X)$. Such a cochain φ is related to ζ in the same manner as a momentum mapping to an equivariant closed 2-form, a momentum mapping for the 2-form (not necessarily non-degenerate and *not* equivariantly closed) being an equivariantly closed extension.

To explain what this means, suppose that G is a compact Lie group, let X be a G-manifold and, instead of $C^*(X)$, take the de Rham complex $\mathcal{A}(X)$. The Cartan model for the G-equivariant de Rham cohomology of X has the form

$$\mathrm{Hom}^{\tau^{\mathrm{S}'}}(\mathrm{S}'[s^2\mathfrak{g}], \mathcal{A}(X))^G; \tag{8.11.2}$$

here $\mathrm{S}'[s^2\mathfrak{g}]$ is the graded symmetric coalgebra on the double suspension $s^2\mathfrak{g}$, the algebra $\Lambda[s\mathfrak{g}]$ is the graded exterior algebra on the suspension $s\mathfrak{g}$, the twisting cochain $\tau^{\mathrm{S}'} \colon \mathrm{S}'[s^2\mathfrak{g}] \to \Lambda[s\mathfrak{g}]$ is the universal twisting cochain, and $\Lambda[s\mathfrak{g}]$ acts on $\mathcal{A}(X)$ via contraction. We now substitute a closed G-invariant 2-form σ for ζ, a G-equivariant $\mathcal{A}^0(X)$-valued 2-cochain Φ on $\mathrm{S}'[s^2\mathfrak{g}]$ for the $C^0(X)$-valued 2-cochain φ on $\overline{\mathrm{B}}C_*G$, that is, essentially a G-equivariant linear map Φ from $\mathfrak{g}$ to $\mathcal{A}^0(X) = C^\infty(X)$ and, likewise, we substitute the twisting cochain $\tau^{\mathrm{S}'}$ for τ. The identity (8.11.1) then translates to the identity

$$d \circ \Phi = \sigma \cup \tau^{\mathrm{S}'} \colon \mathrm{S}'[s^2\mathfrak{g}] \longrightarrow \mathcal{A}^1(X) \tag{8.11.3}$$

where d is the de Rham differential on X. With the notation $\mu \colon X \to \mathfrak{g}^*$ for adjoint of Φ, we may rewrite (8.11.3) as

$$\sigma(\xi_X, \cdot) = \xi \circ d\mu, \quad \xi \in \mathfrak{g}, \tag{8.11.4}$$

where ξ_X refers to the fundamental vector field on X coming from $\xi \in \mathfrak{g}$, and this is exactly the momentum mapping property.

This recovers the familiar fact that a momentum mapping for a closed G-equivariant 2-form σ (not necessarily non-degenerate and *not* equivariantly closed) is an equivariantly closed extension of σ, that is, $d_2[\sigma]$ is the obstruction to the existence of a momentum mapping for σ.

These substitutions can be given an entirely rigorous meaning; to this end, one has to develop the formalism with $\mathrm{S}'[s^2\mathfrak{g}]$ instead of $\overline{\mathrm{B}}C_*G$. In particular, given a splitting homotopy $h\colon \mathrm{S}'[s^2\mathfrak{g}] \to \mathrm{End}(\mathcal{A}(X))$ for $\tau^{\mathrm{S}'}$, the corresponding adjoint thereof includes a morphism of the kind

$$h^\flat\colon \mathfrak{g} \otimes \mathcal{A}^2(X) \longrightarrow \mathcal{A}^0(X)$$

such that, given a closed 2-form $\sigma \in \mathcal{A}^2(X)$, the association

$$\mathfrak{g} \ni \xi \longmapsto h^\flat(\xi \otimes \sigma) \in \mathcal{A}^0(X) \tag{8.11.5}$$

yields a "comomentum" for σ, i. e. the adjoint thereof is an ordinary momentum mapping for σ.

Thus, given a general topological group of strictly exterior type and a G-action on a space X, *a splitting homotopy h for τ generalizes the concept of momentum mapping in a very strong sense* since it provides, in particular, *a single object which yields, via the association* (8.11.5), *a momentum mapping for every closed 2-cocycle and, furthermore, the correct replacement thereof for cocycles of arbitrary degree*.

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